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Analysis and Testing of the SURE Program

Kelly J. Dotson

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Kelly J. Dotson
Langley Research Center
Hampton, Virginia

National Aeronautics and Space Administration

Scientific and Technical Information Division

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Introduction

Reliability of computer systems is a major concern, especially for systems used in life-critical applications. Accurate estimation of reliability is particularly relevant to fault-tolerant computer systems that are often used in such life-critical applications as flight-control systems. Fault-tolerant computer systems utilize such techniques as self-checking codes, redundancy, and reconfiguration to mask the effects of errors in the system in an effort to achieve higher reliability. Analytic computation of the exact reliability of a computer system is virtually impossible except for the most simple systems. Hence, several methods including fault-tree analysis, Petri net analysis, and Markov modeling have been used to estimate the reliability of the hardware components of fault-tolerant computer systems.

As fault-tolerant systems grow in complexity, assessing reliability becomes more difficult. Realistic models of fault-tolerant computer systems can easily have several thousand states. To aid in the reliability assessment of such systems, automated tools—such as CARE III (Computer-Aided Reliability Estimation, version 3), ARIES (Automated Reliability Interactive Estimation System), and PAWS (Padé Approximation With Scaling)—that implement a range of techniques for estimating reliability have been created (refs. 1 and 2). The Semi-Markov Unreliability Range Evaluator (SURE) program developed at NASA Langley Research Center is one of the latest reliability analysis tools to be introduced (ref. 3). The SURE program implements mathematics developed by White (ref. 4) and Lee (ref. 5) for analytically specifying lower and upper bounds on the death-state probabilities of a semi-Markov model in order to provide bounds on the unreliability of a modeled system.

If tools such as SURE are to be used in the assessment of highly reliable computer systems, the tools themselves must produce accurate outputs. For our purposes, a reliability estimate can be defined as accurate if that estimate lies within acceptable limits of the true reliability. Engineering judgment is then used to specify the acceptable limits based on a given system and its intended application. As a general rule of thumb, if a reliability estimate agrees to at least two significant digits with another estimate that is known to be accurate, then the new estimate is considered good enough.

Investigation of the accuracy of SURE's bounds involves demonstrating that the bounds given by SURE actually envelop the exact unreliability for any given semi-Markov model of a system and that the separation between the upper and lower bounds is within an acceptable tolerance. Again, this tolerance is subject to engineering judgment about the system being analyzed and its intended application. To build confidence in the validity of the SURE program, the mathematics has been scrutinized from a theoretical standpoint, and the program has been subjected to numerous models and test cases to inspect the implementation of the mathematics.

White (ref. 4) gives a rigorous, mathematical proof of his multiple recovery theorem that contains the bounds that are the basis of the SURE program. Both the theory and the proof have been subjected to substantial peer review. Specifically, the theory has been reviewed by several mathematicians and has been published in a journal (ref. 6); no flaws have yet been found. Thus, the bounding theory in SURE has been adequately proven correct on a theoretical level.[1]

This report gives the results of the first attempt to independently test the SURE program to demonstrate that the bounds are correctly coded. Two major studies were conducted on version 5.2 of the SURE program. First, SURE's bounds were compared with exact analytic solutions for simple semi-Markov models. Second, SURE's bounds were compared with estimates from other reliability analysis tools (CARE III, ARIES, PAWS, and STEM (Scaled Taylor Exponential Matrix)) for more complex models for which analytic derivation of the exact solution is largely infeasible. And finally, the mathematical bounds were analyzed to

[1] SURE differs from some of the latest reliability analysis tools in that its mathematics has all been rigorously proven.

determine the relative difference in the bounds for models that are pure death processes and models with renewal.

Approach to Testing SURE

Since no formal methodology exists for thoroughly testing a reliability analysis tool such as SURE, one needs to examine those attributes of the tool that are most important to the user and important with respect to the application for which the tool is intended. The SURE tool was designed primarily to assess fault-tolerant computer architectures whose applications usually require high reliability. Hence, the validity of the tool itself must be established. That is, there must be a high degree of confidence that the true unreliability of a semi-Markov model of a system falls between the bounds given by SURE. This immediately raises two questions: (1) what approach can be used to test that the SURE bounds encompass the true unreliability of any system modeled with a semi-Markov model, and (2) how do you measure how much confidence you have in a tool?

Since there are an infinite number of possible semi-Markov models, SURE cannot be tested with every model. Exhaustive testing would require that SURE be tested with all models that represent fault-tolerant systems. Again, this is impossible. The approach chosen for this analysis was to test SURE with a number of models that represent constructions used in modeling fault-tolerant systems starting with elementary building blocks and progressing to large, complex models. As more models are used to test SURE and SURE produces accurate bounds for these models, confidence increases in the validity of the program.

To know for certain that SURE's bounds contain the true unreliability of a modeled system, one must know the true unreliability of that system. Deriving the analytic solution for unreliability is virtually impossible for most models. For those models whose analytic solution cannot readily be obtained, other reliability analysis tools (CARE III, PAWS, ARIES, and STEM) can be used. These tools use a variety of approaches to estimating reliability and can be used to get a "best estimate" of the unreliability, which is generally taken as the majority consensus among the reliability tools. Comparing this best estimate with SURE's bounds would then give confidence that SURE's bounds are correct, although it would not guarantee that SURE's bounds are correct.

Thirty-five models that range from basic constructions to complex models of fault-tolerant architectures were chosen to test SURE. Fifteen simple models were selected to be solved analytically, and twenty more complex models that are pure Markov models were chosen to be solved by other reliability analysis programs to compare with SURE. Figure 1 shows the relationship of the models that were chosen for testing to the domain of semi-Markov models. Most existing reliability analysis tools are designed to evaluate pure Markov models, which are a subset of semi-Markov models. Since Markov models only allow exponentially distributed transitions, Markov models are more restrictive than semi-Markov models, which allow transitions with any statistical distribution. The CARE III program does allow the user to specify nonconstant transition rates; however, the mathematics underlying this semi-Markov part of the program has not been rigorously proven. ARIES, PAWS, and STEM are strictly Markov solvers. Hence, deriving the analytic solutions for simple semi-Markov models is currently the only practical means of testing SURE's bounds for semi-Markov models.

Besides selecting an appropriate set of models to test SURE, the program can be stress tested by choice of the parameter values for each model. In general, for fault-tolerant systems the fault-arrival rate is expected to be slow, that is, from 10^{-3} to 10^{-5} faults per hour, and the range for the fault-recovery rate is expected to be fast, that is, from 10^{3} to 10^{5} recoveries per hour. To stress the program, the fault-arrival rates used in the test cases range from 1 to 10^{-8} faults per hour, and the fault-recovery rates range from 10^{-2} to 10^{8} recoveries per hour. Again, exhaustive testing of every combination of parameter values is impossible. The following combinations of parameter values were often chosen since they are thought to be fairly extreme

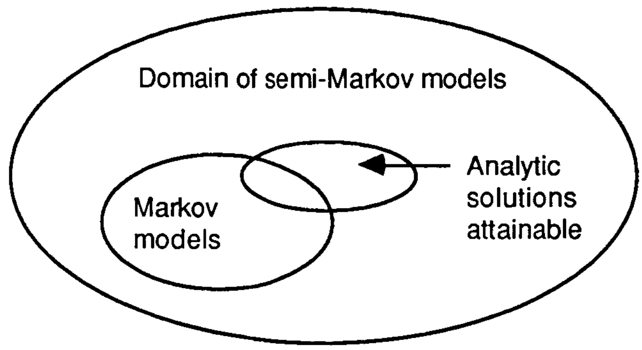

Figure 1. Relationship of test models to the domain of semi-Markov models.

cases that might show weaknesses in the program: fast fault-arrival rates, slow fault-recovery rates, fast fault-arrival rates with slow recovery rates, fast fault-arrival rates with very fast recovery rates, and very slow fault-arrival rates with very fast recovery rates.

The second question dealing with measurement of user confidence in a tool is more difficult to address. Since one cannot exhaustively test the program and there is no capability to prove code of the complexity of SURE, one cannot conclude after any amount of testing that there is 100-percent confidence that SURE's bounds are correct. Confidence in such a program can be measured only from a relative standpoint. As stated earlier, the entire mathematical basis for SURE has been rigorously proven. This fact, in itself, lends much confidence to the correctness of the bounds implemented in the program. The test cases presented in this report along with the analysis of the relative error in the bounds further increase user confidence in SURE.

SURE Program and Computing Exact Unreliability From a Model

In a Markov model of a fault-tolerant system, the unreliability of the system is calculated as the sum of all death-state probabilities in the model. A death state in a Markov model is a state that has no exiting transitions; that is, an absorbing state. In relation to a computer system, a death state often represents the system's failure. Each death-state probability can be determined by individually looking at each path in the model that leads to that death state. The exact death-state probabilities of a semi-Markov model are determined mathematically by solving a series of convolution integrals. The number of convolution integrals increases as the size of the model increases; thus finding an exact solution for a large model is essentially impossible. Because of the difficulty of solving convolution integrals, the models in this first phase of testing were limited to five states.

To calculate a death-state probability, each path that leads to that death state can be analyzed transition by transition. In a pure Markov model, the transitions between states are all exponentially distributed. However, in a semi-Markov model, transitions between states can be described by any statistical distribution. For this work, transition rates were limited to distributions that are mathematically tractable, namely, the exponential, uniform, and impulse distributions.

Review of SURE Mathematics

In SURE, the unreliability of a semi-Markov model is computed by analyzing each possible path through the model from the model's initial, no-failure state to termination at some death state. Since global time independence is an inherent property of a semi-Markov model, the states of the model can be rearranged to facilitate estimation without changing the numerical result. The mathematics of SURE employs this strategy. The SURE program assesses the unreliability by arranging the path steps of the entered model into three different path classifications.

The paths are classified on the basis of whether the relative rates of the transitions between each of the states are slow or fast. Slow transition rates correspond to fault arrivals in a computer system and are assumed to be exponentially distributed in the program. Fast rates describe the system's response to faults and can be characterized by any distribution given the distribution's mean and variance. The slow transitions are denoted in the models by a lowercase Greek character (λ, γ, ε, δ, α, and β) representing the rate; the general transitions are denoted by a capital letter (F, G, and H) that represents a particular distribution.

A path step consists of a state and each of the transitions leaving it. For a particular path through a model, the transitions directly on the path being analyzed are called the on-path transitions and the remaining transitions are referred to as off-path. For any given path step, there can be only one on-path transition; but, there can be arbitrarily many off-path transitions. To illustrate the concept of on- and off-path transitions, consider the model in figure 2. In this model there are two possible paths that lead to state 6: $1 \rightarrow 2 \rightarrow 3 \rightarrow 5 \rightarrow 6$ and $1 \rightarrow 2 \rightarrow 4 \rightarrow 5 \rightarrow 6$. Consider the path $1 \rightarrow 2 \rightarrow 4 \rightarrow 5 \rightarrow 6$. The on-path transitions in this path are the transitions between states 1 and 2, 2 and 4, 4 and 5, and 5 and 6. The transition between states 2 and 3 is an off-path transition from state 2.

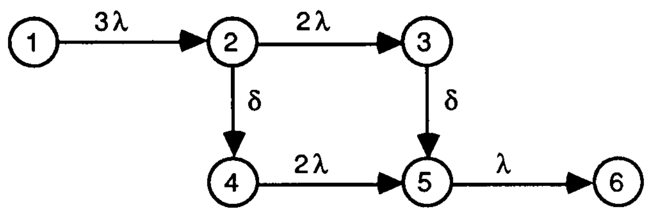

Figure 2. Markov model for demonstration of on- and off-path transitions.

Three path-step classifications based on the state transition rates are used in SURE. These are (1) slow on-path, slow off-path, (2) fast on-path, arbitrary off-path, and (3) slow on-path, fast off-path. Figures 3, 4, and 5 are examples of the path-step classifications given in reference 3.

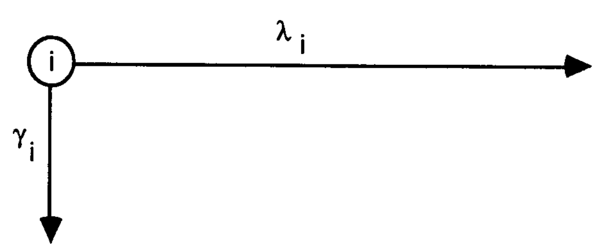

Figure 3. Class 1 path step: slow on-path, slow off-path transitions.

In figure 3, λ_i represents the rate of the on-path transition from state i, and γ_i represents the sum of all the slow off-path transitions from state i. No fast off-path transitions are allowed in the class 1 category. In figure 4, $F_{i,k}$ where $k = 1, 2, ..., n_i$ represents the kth fast transition from state i, and n_i represents the number of fast transitions leaving state i. The sum of all the slow transitions leaving state i is given by ε_i . In SURE, three parameters must be specified for each general transition. The first two parameters for the class 2 path step are the conditional mean $\mu(F^*_{i,k})$ and conditional variance $\sigma^2(F^*_{i,k})$, given that the general transition $F_{i,k}$ occurs.

The third parameter is the transition probability $\rho(F^*_{i,k})$, the probability that the transition from state i to k occurs. The following equations define these parameters:

$$\rho(F^*_{i,k}) = \int_0^\infty \prod_{j \neq k} [1 - F_{i,j}(t)] \; dF_{i,k}(t)$$

$$\mu(F^*_{i,k}) = \frac{1}{\rho(F^*_{i,k})} \int_0^\infty t \prod_{j \neq k} [1 - F_{i,j}(t)] \; dF_{i,k}(t)$$

$$\sigma^2(F^*_{i,k}) = \left\{ \frac{1}{\rho(F^*_{i,k})} \int_0^\infty t^2 \prod_{j \neq k} [1 - F_{i,j}(t)] \; dF_{i,k}(t) \right\} - \mu^2(F^*_{i,k})$$

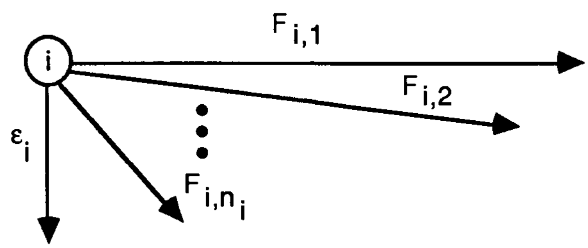

Figure 4. Class 2 path step: fast on-path, arbitrary off-path transitions.

where t is time.

In the class 3 category in figure 5, the on-path transition α_j is slow as in class 1; but, there are also n_j fast off-path transitions leaving state j represented by $G_{j,k}$ where $k = 1, 2, ..., n_j$. Slow off-path transitions may or may not be present. As in the other path classifications, all

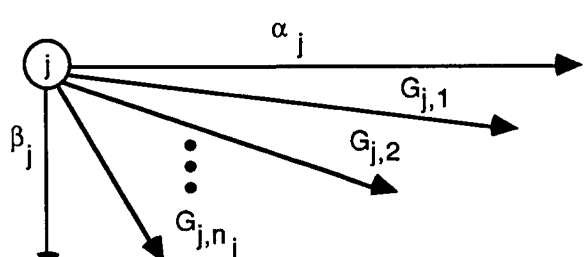

Figure 5. Class 3 path step: slow on-path, fast off-path transitions.

the slow off-path transitions are represented by a sum, given by β_j. The parameters needed to describe the general transitions in this path step are defined in the same manner as those for the class 2 path step:

$$\rho(G^*_{j,k}) = \int_0^\infty \prod_{i \neq k} [1 - G_{j,i}(t)] \; dG_{j,k}(t)$$

$$\mu(G^*_{j,k}) = \frac{1}{\rho(G^*_{j,k})} \int_0^\infty t \prod_{i \neq k} [1 - G_{j,i}(t)] \; dG_{j,k}(t)$$

$$\sigma^2(G^*_{j,k}) = \left\{ \frac{1}{\rho(G^*_{j,k})} \int_0^\infty t^2 \prod_{i \neq k} [1 - G_{j,i}(t)] \; dG_{j,k}(t) \right\} - \mu^2(G^*_{j,k})$$

The three path steps are each combined to form paths as shown in figure 6, where each path step in each path is of the same type. These paths can then be combined to form a general path.

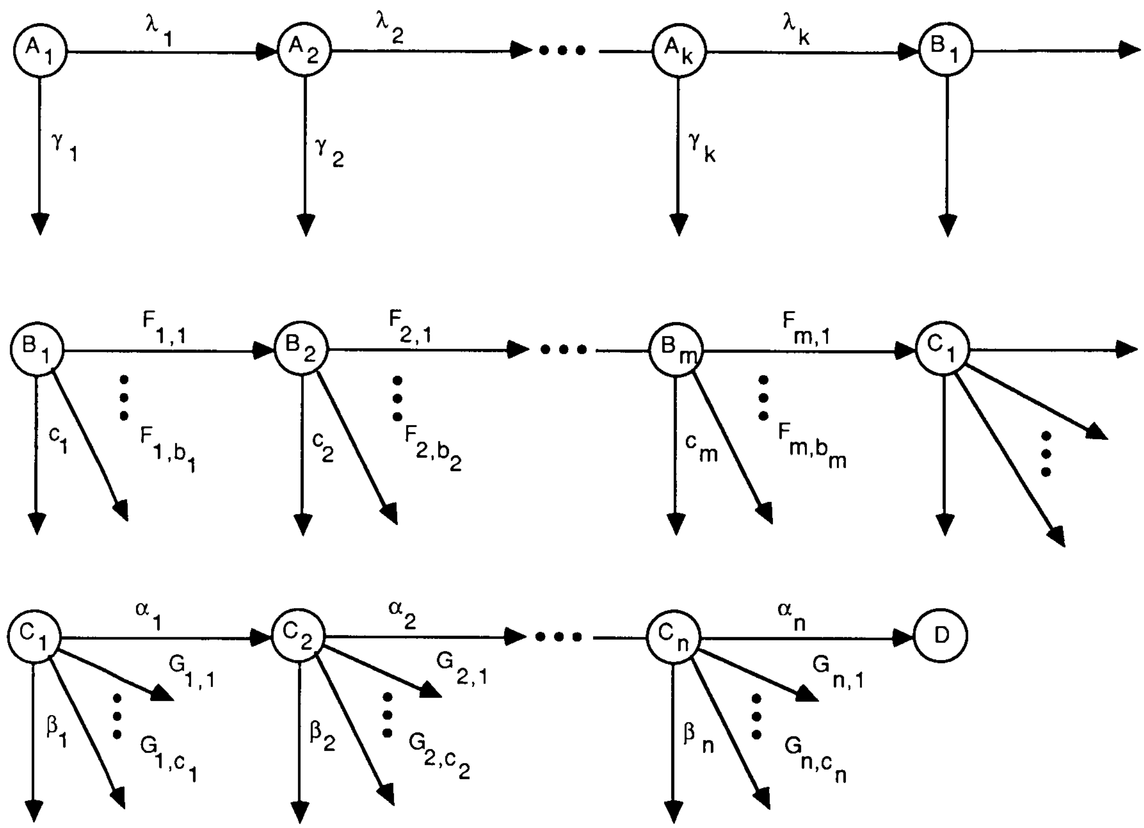

Figure 6. General path consisting of the three path-step classifications.

In figure 6, the path steps from A_k are class 1, those from B_m are class 2, those from C_n are class 3, and state D represents a death state, a state from which there are no exiting transitions. The unreliability of the modeled system at time T is the sum of the probabilities of being in each of the death states at T. The probability of being in a death state at the end of each of the above path classifications, denoted $P_i(T)$ where $i = 1$, 2, or 3 for each classification, is defined as follows:

$$P_1(T) = \int_0^T \int_0^{T-x_1} \cdots \int_0^{T-x_1-\cdots-x_n} \lambda_1 e^{-(\lambda_1+\gamma_1)x_1} \lambda_2 e^{-(\lambda_2+\gamma_2)x_2} \cdots$$
$$\times \lambda_k e^{-(\lambda_k+\gamma_k)x_k} dx_k \cdots dx_1$$

$$P_2(T) = \int_0^T \int_0^{T-x_1} \cdots \int_0^{T-x_1-\cdots-x_n} e^{-\varepsilon_1 x_1}[1 - F_{1,2}(x_1)] \cdots [1 - F_{1,b_1}(x_1)] \cdots$$

$$\times\, e^{-\varepsilon_m}[1 - F_{m,2}(x_m)] \cdots [1 - F_{m,b_m}(x_m)] dF_{m,1}(x_m) \cdots dF_{1,1}(x_1)$$

$$P_3(T) = \int_0^T \int_0^{T-x_1} \cdots \int_0^{T-x_1-\cdots-x_n} \alpha_1 e^{-(\alpha_1+\beta_1)y_1}[1 - G_{1,2}(y_1)] \cdots$$

$$\times\, [1 - G_{1,c_1}(y_1)] \cdots \alpha_n e^{-(\alpha_n+\beta_n)y_n}[1 - G_{n,1}(y_n)] \cdots [1 - G_{n,c_n}(y_n)] dy_n \cdots dy_1$$

To compute the death-state probability from a path that contains a combination of classifications, the integrals are simply combined as in White's synthetic bounds (ref. 4, pp. 7–8). Using these formulas, one can calculate the unreliability of a simple semi-Markov model.

Comparison of SURE With Analytic Solutions—Phase I

During the first phase in the testing of SURE, the bounds developed by White (ref. 4) were examined. Since the exact unreliability can be determined for a very simple semi-Markov model of a system, the first step in testing the accuracy of SURE was to compare SURE's bounds using version 5.2 of the program with exact solutions. For this phase of testing, 15 very simple semi-Markov models were constructed, and the exact unreliability was analytically derived for each. Each model was then run several times, each time varying the value of the parameters of the model in order to stress the program and identify parameter ranges where the bounds separate or the program fails. A series of test cases, each consisting of the exact solution, SURE's bounds, and a relative difference estimate, are presented for each model.

The relative difference estimate, denoted RD, given for the bounds is defined as follows:

$$\mathrm{RD} = \frac{|\text{SURE bound furthest from exact solution} - \text{Exact solution}|}{\text{Exact solution}}$$

This difference estimate gives a measure of the tightness of the bounds. The unreliability estimate given by SURE is more precisely expressed when the bounds are tight. A small relative difference indicates tight bounds. Correspondingly, a large relative difference indicates a wide spread in the bounds. The acceptable degree of tightness in the bounds is measured by engineering judgment that is based on the needs of the user and the intended application.

Appendix A contains the results of this testing. For each example, the model is given along with the equations for the exact death-state probabilities. Equations for μ, σ^2, and ρ are given for models with nonexponential transition rates. The correctness of these equations was checked with MACSYMA (ref. 7), a computer programming system developed at Massachusetts Institute of Technology. MACSYMA applies a symbolic manipulation approach to processing mathematical expressions. A table that compares the exact solutions with the SURE bounds for a range of parameter values accompanies each model.

The SURE program does not require the user to specify units when inputting the model's transition parameters. Since reliability is measured with respect to a specified length of time T, referred to here as "mission time," the user must be consistent in the specification of the mission time and transition rates. For example, if the reliability of a model for a 10-hour mission is desired, then each of the transition rates should be given in terms of hours. For all cases reported in this study, the mission time default of 10 hours, that is, T=10, was used unless specified otherwise. To be consistent, the unit of measure for all the transition rates is assumed to be hour^{-1}.

Because of numerical stability problems, a few of the analytic solutions were calculated using Taylor series expansion techniques. Since the SURE bounds are given in six significant digits, the exact solutions were also rounded off to six significant digits. All warning and error messages that were output by the SURE program when each case was run are noted at the bottom of the appropriate table and are explained in the discussion of the results.

Although the test cases shown in appendix A were elementary, they do represent basic constructions used in modeling fault-tolerant computer systems. For all these test cases, the exact unreliability for a given model was always enclosed by SURE's bounds. In most of the cases, the upper bound was much closer to the exact solution than the lower bound. When estimating the unreliability of a computer used in life-critical applications, having a conservative estimate of the unreliability is important. Since the upper bound is a conservative estimate, the upper bound serves as a good estimator of the unreliability for most of the cases examined. In general, the bounds were also very tight. The relative difference was less than 5 percent in 71 percent of the cases and even less than 1 percent in many of these cases.

There were, however, certain parameter values that caused the bounds to separate; in a few cases, the bounds would not even provide a useful estimate of unreliability. The wide bounds can largely be attributed to general transition rates that are slow with respect to the mission time, that is, slow fault-recovery rates. The general transition rates that cause bound separation can be characterized by looking at the mean of the general transition distribution and the mission time. When the product of the mean of the general transition and the mission time is greater than 5, the bounds tend to separate significantly; that is, the relative difference is usually greater than 5 percent. When this product is greater than 30, a warning message "RECOVERY TOO SLOW" is output by SURE. SURE also warns the user if the mean of the general transition is greater than the mission time by issuing the message "DELTA > TIME."

Separation of the bounds also occurred when a fast exponential rate was expressed as a slow transition. When a slow exponential rate was greater than or equal to 0.01/hour, that is, the fault-arrival rate was fast, there tended to be a moderate separation of the bounds. The test cases for examples 4a and 4b in appendix A demonstrated the effect of describing a fast exponential transition with the construct for a slow transition. To yield tighter bounds, fast exponential transitions must be specified as general transitions with mean and standard deviation. The warning messages "RATE TOO FAST" and "DELTA > TIME" were output when fast transitions were not correctly specified.

As mentioned earlier, the user must decide based on the intended application whether or not the bounds given by SURE are tight enough to meet the application's requirements. One should keep in mind when using SURE that the mathematics implemented in the program was designed to solve semi-Markov models that describe the failure behavior of highly reliable, reconfigurable, fault-tolerant systems which exhibit slow fault-arrival rates and very fast recovery rates. The fast fault-arrival rates and slow fault-recovery rates that induced the separated bounds are not consistent with such systems. Separation of the bounds is also demonstrated in the second phase of testing where model complexity precludes use of analytic solutions. In these more complex cases, the SURE program is found to be well behaved as its application limits are approached. In the second phase, described in the next section, more complex models and models that represent actual fault-tolerant architectures are used to test the SURE program.

Comparison of SURE With Other Reliability Analysis Tools—Phase II

The next step in the investigation concentrated on testing the program with more complex models and models that represent actual fault-tolerant computer systems. To estimate the reliability of state-of-the-art fault-tolerant architectures, tools such as ARIES, CARE III, PAWS, and STEM have been developed. These programs are described in more detail in the following paragraphs. In general, these tools apply different numerical, aggregation, and decomposition techniques to estimate the reliability from a model of a given system. Since these tools were developed to assess fault-tolerant systems where reliability is a key issue, the estimates for reliability should be conservative. Although comparison of SURE's bounds with these estimates does not guarantee that the bounds actually envelop the exact unreliability, this comparison at least offers the user a sense of confidence in SURE's bounds.

Description of CARE III, ARIES, PAWS, and STEM

The CARE III reliability analysis tool uses a solution technique that applies behavioral decomposition and aggregation methods to evaluate the reliability of fault-tolerant computer systems (ref. 8). The solution technique assumes that fault occurrence is a relatively infrequent (low-rate) event, and fault-handling behavior is composed of relatively frequent (high-rate) events. CARE III reduces the solution of a complex model to the solution of two relatively simpler models: a coverage model that is a semi-Markov process and a reliability model that is a nonhomogeneous, Markov process (ref. 9). Numerical integration techniques are then used to solve these Markov models. The solution technique involves an approximation that is not characterized via a mathematical error analysis. Fault trees are used to describe the fault-occurrence behavior, and the fault-handling behavior is described by the parameters of the semi-Markov model. The CARE III program was originally developed at the Raytheon Company and has since been modified and enhanced. Version 6 of the program was used for this investigation.

The ARIES program (ref. 10) evaluates the reliability of systems that are defined as homogeneous Markov processes. This program was developed at the University of California at Los Angeles as an interactive reliability modeling tool. A decomposition technique is implemented in ARIES in the sense that the system being analyzed is defined as a series of subsystems where each subsystem is separately analyzed. By approximating fault-handling states with instantaneous coverage, the state size of systems is also reduced. The ARIES program incorporates six models that can describe closed, repairable, and renewable systems. ARIES does not provide a comprehensive error analysis with its reliability estimates. For examples 29 through 35, the ARIES 82 version was used.

The PAWS program, which was developed at NASA Langley Research Center, is used to compute the reliability of a pure Markov model. The reliability of a system modeled with a pure Markov model can be determined by solving a system of differential equations. PAWS uses a combination of Padé approximations, scaling, and squaring techniques to compute a matrix exponential needed to solve this system of equations and, hence, determine the death-state probabilities of a Markov model. This method of finding the matrix exponential is considered one of the most efficient algorithms known (ref. 11). A conservative estimate of the number of digits of accuracy in the unreliability estimate is also given along with the death-state probabilities. PAWS is limited, though, to pure Markov models and cannot handle very large models (models with more than 300 states). Enumeration of each state transition is needed for PAWS, and PAWS uses the same input format as SURE.

Another reliability analysis tool called STEM, which was also developed at NASA Langley Research Center, does have the capability to compute the exact death-state probabilities for Markov models with up to 1000 states. The underlying mathematics in STEM involves the calculation of the matrix exponential, which is defined via a Taylor series (refs. 12 and 13). The Taylor series is truncated in the program, and a conservative error estimate of the truncation is produced. The STEM program uses the same input language as SURE and outputs the death-state probabilities along with the error estimate.

Description of Models for Phase II

During the second phase of the testing, SURE's bounds were compared with unreliability estimates given by CARE III, PAWS, STEM, and ARIES for a set of models. Since most of the reliability analysis tools apply only to pure Markov processes, only models with exponential transitions were considered. The models used in this phase of testing exhibit more complex constructs than the models in the first phase. Some of the models include permanent, transient, and intermittent faults.

Models with renewal (transient or intermittent faults) have an infinite number of paths of increasing lengths leading to death states. Such a model is shown in figure 7, where γ is the transient-fault-disappearance rate and α and β are intermittent-fault rates. Theoretically, an infinite number of paths would have to be analyzed to compute the bounds. However, as a path grows longer, the probability of entering a death state diminishes. For any long path, there is generally a point along that path where the contribution to the overall unreliability of the modeled system becomes insignificant. Truncation and pruning techniques can then be implemented to analyze renewal models.

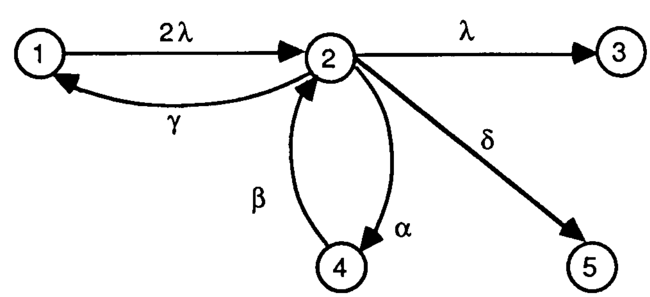

Figure 7. Renewal model with transient and intermittent faults.

With the truncation feature implemented in SURE, the number of times a loop is unfolded can be specified. The pruning feature allows a value to be specified so that processing of the path stops once the probability of going down that path drops below the designated value. Upper bounds on the error introduced by truncation and pruning are given in SURE's output. Thus, a model that actually contains an infinite number of paths can be analyzed as if it has a finite number of paths. Use of the truncation and pruning techniques allows White's bounds to be applied to models with renewal. A more detailed discussion of truncation and pruning can be found in reference 3.

The models used in this phase of testing are found in appendix B. Twenty models were considered, and these were divided into two sets. The models in the first set, examples 16 to 28, are still fairly simple models compared with the complexity of most real fault-tolerant systems. These models ranged in size from 5 to 450 states. Although several of the models are relatively simplistic, they do represent realistic models of fault-tolerant architectures. The SURE program was run for each of these models and compared with estimates from CARE III (where it applied), PAWS, and STEM (in one case with a particularly large model). The Markov model, or part of the model, is given for each example followed by a table that shows each tool's unreliability estimates.

All the default run-time parameters of the CARE III program were used unless otherwise noted. Some of the systems could not be directly modeled with CARE III, and these situations are noted where appropriate. In particular, CARE III cannot directly model systems that consider critical-triple faults or cold spares. It should also be noted that the CARE III user's guide warns against using some of the parameter values that are found in some of the following test cases. These restrictions are explained in greater detail where they apply.

The last set of models, examples 29 through 35 located in appendix C, was taken from reference 14. This report describes a comparative analysis between the ARIES and CARE III tools designed to show the strengths and weaknesses of each tool for analyzing architectures for fault-tolerant aerospace systems. In the report, seven simple reliability models were analyzed with ARIES and CARE III and compared with a direct calculation of the unreliability of the modeled system. The SURE program was run for each of these models, and SURE's bounds are given in appendix C along with the corresponding results from ARIES, CARE III, and a direct calculation of the unreliability. (The direct calculations and analytic approximations shown in this report are accurate calculations of the unreliability of the modeled system. Some of the direct calculations given in reference 14 are not correct. The ARIES and CARE III programs were not run again to check that the results in the reference were accurate.)

Summary of Results of Phase II Testing

During the second phase of testing the accuracy of SURE's bounds, no cases were found where the final bounds given by SURE did not cover the best estimates of the unreliability of the modeled systems. As seen during the first phase, the upper bound was usually a much better estimator of the unreliability than the lower bound. In most cases considered in this phase, the bounds on the total unreliability were extremely tight. In fact, the bounds in each of the cases agreed with the best estimate in the order of magnitude of the total system's unreliability. There were a small number of cases, however, where the bounds were significantly separated.

Separation of the bounds occurred when the fault-arrival rate was relatively fast with respect to the mission time or the fault-recovery rate was relatively slow. These rates are not typical of highly reliable, fault-tolerant computer systems. The separation of the bounds in these cases was also not as severe as in the cases where nonexponential transitions were used. That is, the bounds seem to be tighter for models of pure Markov processes than semi-Markov processes.

Separation of the bounds was also more apparent for models with transient and intermittent faults. As mentioned previously, White's bounds were not explicitly designed for models with renewal. In the models where only permanent and transient faults were present, the bounds tended to converge quickly. In only a few cases was a truncation level of more than three required to yield good bounds. If the truncation level was set small enough that the bounds had not converged sufficiently, a "TRUNC TOO SMALL" warning message was issued. In such cases, the model should be run again with a higher truncation level until the warning message disappears.

For the models with intermittent faults, however, both truncation and pruning were necessary in some of the cases to produce good bounds. As in the case of truncation, if the pruning level was set too large to ensure correct bounds, a "PRUNING TOO SEVERE" message was issued. When this warning message was issued, the bounds did not necessarily envelop the true unreliability of the modeled system; and the model was run again with a new pruning level until the warning disappeared.

There were a few cases where pruning was used and the sum of the pruned states' probability was significantly larger than the true unreliability. This sum was given as part of SURE's output so that the user could be aware of the error caused by pruning; however, no warning message was given to draw the user's attention to the fact that the bounds may not be accurate. When this sum is larger than, or within a couple of orders of magnitude of, the given bounds, the bounds may not enclose the true unreliability because of the error caused by pruning. Thus, when the prune feature is used, the user should be mindful of the value given at the bottom of SURE's output that gives an upper bound on the sum of the pruned states' probability. If this sum is within a couple of orders of magnitude of the bounds currently given, the model should be run with a smaller prune value to ensure sufficient convergence of the bounds.

In the following section, the relative errors in the bounds for models with intermittent and transient faults are derived. These error estimates help demonstrate the rate of convergence of the bounds in these cases.

Analysis of SURE's Mathematical Bounds—Phase III

While SURE was being exercised with various models, no cases were found where SURE's final bounds did not contain the unreliability for any model tested. However, there were cases in which the bounds were so widely separated that they were essentially useless for describing the unreliability of a system. During this phase in the investigation of the validity of SURE's bounds, we look at the algebraic bounds themselves to discern those parameter values that cause the bounds to separate. To do this, relative error estimates for three different types of models were considered: (1) models of pure death processes, (2) models with transient faults, and (3) models with intermittent faults.

Error Analysis for General Model

The mathematical bounds developed by White were examined to determine a relative error estimate in the bounds for models that are pure death processes and for models that are renewal processes. White's bounds were originally intended for pure death processes. For a pure death process, the exact difference between SURE's upper and lower bounds can be determined. A sensitivity analysis of the relative error in pure death cases can show those parameters that lead to separation in the bounds. From White's multiple recovery theorem, the upper bound, denoted UB, and the lower bound, denoted LB, for the unreliability of a semi-Markov model are given as follows (for notation, refer back to figs. 3–5 on path classification):[2]

$$
\begin{aligned}
\mathrm{UB} &= Q(T)\prod_{i=1}^{m}\rho(F_i^*)\prod_{j=1}^{n}\alpha_j\mu(H_j) \\
\mathrm{LB} &= Q(T-\Delta)\prod_{i=1}^{m}\rho(F_i^*)\left[1-\varepsilon_i\mu(F_i^*)-\frac{\mu^2(F_i^*)+\sigma^2(F_i^*)}{r_i^2}\right] \\
&\quad\times\prod_{j=1}^{n}\alpha_j\left[\mu(H_j)-(\alpha_j+\beta_j)\frac{\mu^2(H_j)+\sigma^2(H_j)}{2}-\frac{\mu^2(H_j)+\sigma^2(H_j)}{s_j}\right]
\end{aligned}
$$

for all values of $r_i > 0$ and $s_j > 0$, where m is the number of general transitions in the class 2 path step, n is the number in the class 3 path step, and

$$\Delta = r_1 + \cdots + r_m + s_1 + \cdots + s_n$$

where $Q(T)$ is the probability of traversing a path consisting of only the class 1 path steps within time T. The following are lower and upper bounds on $Q(T)$ that are implemented in SURE:

$$
\begin{aligned}
Q(T) &> \frac{\lambda_1\lambda_2\cdots\lambda_k T^k}{k!}\left(1-T\sum_{i=1}^{k}\frac{\lambda_i+\gamma_i}{k+1}\right) \\
Q(T) &< \frac{1}{|S|!}\prod_{i\in S}\lambda_i T
\end{aligned}
$$

where k is the number of class 1 path steps, $S=\{i|\lambda_i T<1\}$, and $|S|$ is the cardinality of the set S. In the SURE program, r_i and s_j default to the following values:

[2] The quantities $\mu(H_j)$ and $\sigma^2(H_j)$ in the equations for UB and LB refer to a holding-time distribution. Details of calculating these parameters are not relevant to the error calculations and are omitted but can be found in ref. 3.

$$r_i = \left\{2T[\mu^2(F_i) + \sigma^2(F_i)]\right\}^{1/3}$$

$$s_j = \left\{\frac{T[\mu^2(H_j) + \sigma^2(H_j)]}{\mu(H_j)}\right\}^{1/2}$$

In reference 3, Butler shows that these values of r_i and s_j are nearly optimal in that they have been found to give very close bounds in most cases.

To study the sensitivity of the bounds to the parameter values, consider the relative error estimate RE where RE = (UB − LB)/UB. Substituting the previous expressions for UB and LB as shown above and simplifying gives

$$\mathrm{RE} = \left\{Q(T)\prod_{j=1}^{n}\mu(H_j) - Q(T-\Delta)\prod_{i=1}^{m}\left[1 - \varepsilon_i\mu(F_i^*) - \frac{\mu^2(F_i^*) + \sigma^2(F_i^*)}{r_i^2}\right]\right.$$
$$\left.\times\prod_{j=1}^{n}\left[\mu(H_j) - (\alpha_j + \beta_j)\frac{\mu^2(H_j) + \sigma^2(H_j)}{2} - \frac{\mu^2(H_j) + \sigma^2(H_j)}{s_j}\right]\right\}\Bigg/ Q(T)\prod_{j=1}^{n}\mu(H_j)$$

To simplify this expression even further, define the variables z_i and y_j as

$$z_i = 1 - \varepsilon_i\mu(F_i^*) - \frac{\mu^2(F_i^*) + \sigma^2(F_i^*)}{r_i^2}$$

$$y_j = \mu(H_j) - (\alpha_j + \beta_j)\frac{\mu^2(H_j) + \sigma^2(H_j)}{2} - \frac{\mu^2(H_j) + \sigma^2(H_j)}{s_j}$$

From these expressions and the upper and lower bounds on $Q(T)$, RE can be rewritten as

$$\mathrm{RE} = 1 - \frac{(T-\Delta)^k\left[1 - (T-\Delta)\sum_{i=1}^{k}(\lambda_i + \gamma_i)/(k+1)\right]\prod_{i=1}^{m} z_i \prod_{j=1}^{n} y_j}{T^k \prod_{j=1}^{n}\mu(H_j)}$$

Since one would expect the parameters $\lambda_i, \varepsilon_i, \alpha_j$, and β_j to be on the order of 10^{-4}/hour and μ and σ to be on the order of 10^{-4} hour in a fault-tolerant computer system, the relative error should be small. In reference 3, Butler gives $k\Delta/T$ as an approximation for this relative error. This approximation is fairly accurate when compared with the exact relative error; however, in most cases $k\Delta/T$ is not a conservative estimate of the relative error. This approximation is especially poor when T is large since $k\Delta/T$ decreases as T increases, while RE increases as T increases.

By examining the relative error, particular ranges of the parameter values become evident which cause the bounds to separate. The relative error tends to increase significantly as the mission time T increases, especially as T increases relative to λ. Large values of λ, as were seen in all the test cases, also cause the bounds to separate significantly. In particular, the combination of a large mission time with a fast fault-arrival rate causes severe separation in the bounds. Slow recovery times (μ large) also cause an increase in the relative error. The relative error is also sensitive to the standard deviation σ being larger than the mean μ. The relative error increases as σ increases with respect to μ. The parameters ε, α, and β seem to affect the relative error very little.

When the parameter values are small (less than 10^{-4}), though, the relative error is generally small. Thus, the bounds should be relatively tight for models of pure death processes when the parameter values reflect those of a fault-tolerant system—slow fault-arrival rates and fast fault-recovery rates.

By applying model-truncation and model-pruning techniques, White's theorem can also be applied to models that contain renewal processes. The relative error in the bounds for models with renewal, however, can only be estimated; but a conservative estimate can be found. Through this relative error estimate, the convergence of models with transient and intermittent faults can be shown.

Error Analysis for Transient-Fault Model

First consider the model in figure 8 that allows for transient faults in the modeled system.

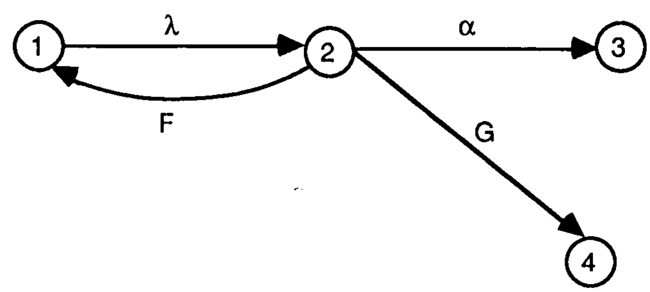

Figure 8. Semi-Markov model with transient faults.

The probability of being in state 3 at time T after N passes through the loop is given by $P_3^N(T)$ where

$$P_3^N(T) = \frac{\alpha\mu(H)\rho^N(F)(\lambda T)^{N+1}}{(N+1)!}$$

The exact probability of being in state 3 at time T is

$$P_3(T) = \sum_{N=0}^{\infty} P_3^N(T) = \sum_{N=0}^{\infty} \frac{\alpha\mu(H)\rho^N(F)(\lambda T)^{N+1}}{(N+1)!} = \frac{\alpha\mu(H)[e^{\rho(F)\lambda T} - 1]}{\rho(F)}$$

If $\rho(F)\lambda T$ is small (this quantity should be small especially for short mission times), then the quantity $\rho(F)(\lambda T)^{N+1}/(N+1)!$ approaches 0 very quickly as N increases. So, a good approximation for $P_3(T)$ can be obtained when $\rho(F)\lambda T$ is small with only a few passes through the loop. The exact error involved in truncating the path after N passes through the loop is given by

$$E_N = P_3(T) - \sum_{i=0}^{N} P_3^i(T) = \frac{\alpha\mu(H)}{\rho(F)} \sum_{j=N+2}^{\infty} \frac{[\rho(F)\lambda T]^j}{j!}$$

From Taylor's theorem,

$$\sum_{j=N+2}^{\infty} \frac{[\rho(F)\lambda T]^j}{j!} \leq e^{\rho(F)\lambda T} \frac{[\rho(F)\lambda T]^{N+2}}{(N+2)!}$$

So, the percentage error $\%E_N$ in the bounds due to truncating a path after N passes through the loop is defined as follows:

$$\%E_N = \frac{E_N}{P_3(T)} 100\% \leq \frac{e^{\rho(F)\lambda T}[\rho(F)\lambda T]^{N+2}/(N+2)!}{e^{\rho(F)\lambda T} - 1} 100\%$$

Thus, $\%E_N \approx \left\{e^{\rho(F)\lambda T}[\rho(F)\lambda T]^{N+1}/(N+2)!\right\} 100\%$ for $\rho(F)\lambda T$ small. This expression for the error vividly shows that the tightness of the bounds for models with transient faults heavily depends on the parameters $\rho(F), \lambda$, and T. Table 1 gives the approximate percentage error in the bounds for a renewal model with transient faults for a range of values of $\rho(F)\lambda T$ and N. As seen in this table, the error involved in truncating a path as in figure 8 after N passes through the loop is generally very small. When the mission time is large or the fault-arrival rate is particularly fast, SURE's truncation parameter should be increased. (Note that the SURE program defaults to three passes through a loop.)

Table 1. Percentage Error in SURE's Bounds for Transient-Fault Models

	Percentage error in SURE's bounds			
$\rho(F)\lambda T$	$N = 2$	$N = 3$	$N = 5$	$N = 10$
0.9	7.471	1.345	2.594×10^{-2}	2.127×10^{-5}
.5	8.587×10^{-1}	8.587×10^{-2}	5.111×10^{-4}	2.218×10^{-8}
.1	4.605×10^{-3}	9.210×10^{-5}	2.193×10^{-8}	3.046×10^{-16}
.01	4.208×10^{-6}	8.417×10^{-9}	2.004×10^{-14}	2.783×10^{-27}
.001	4.171×10^{-9}	8.342×10^{-13}	1.986×10^{-20}	2.758×10^{-38}
.0001	4.167×10^{-12}	8.334×10^{-17}	1.984×10^{-26}	2.756×10^{-49}
.00001	4.167×10^{-15}	8.334×10^{-21}	1.984×10^{-32}	2.756×10^{-60}

Error Analysis for Intermittent-Fault Model

Now, consider a path from a model that contains intermittent faults as in figure 9. An intermittent fault can fluctuate between an active and nonactive state, and $Q(t)$ represents the holding time in the benign state which is state 5 in the model. From White's upper bound the probability of being in state 3 at time T is given as

$$P_3(T) \approx \alpha\lambda\mu(H_2)T[1 + \rho(F) + \rho^2(F) + \cdots] = \frac{\alpha\lambda\mu(H_2)T}{1 - \rho(F)}$$

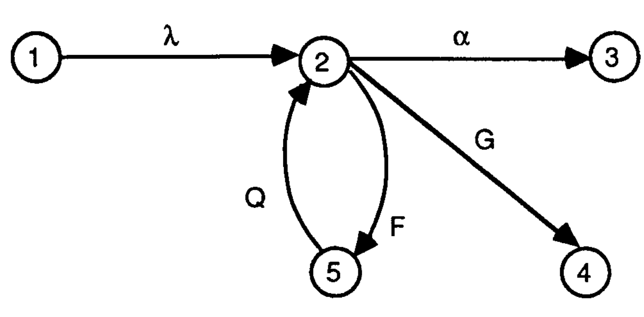

Figure 9. Semi-Markov model with intermittent faults.

And the probability of being in state 3 after N passes through the loop is

$$P_3^N(T) \approx \alpha\lambda\mu(H_2)T\rho^N(F)$$

The error in truncating the loop in figure 9 after N passes is

$$E_N = P_3(T) - \sum_{j=0}^{N} P_3^j(T) = \frac{\alpha\lambda\mu(H_2)T\rho^{N+1}(F)}{1-\rho(F)}$$

From this equation, the rate of convergence of $P_3(T)$ obviously heavily depends on $\rho(F)$. To demonstrate this dependence, suppose that both the F and G transitions are fast exponentials so that $\rho(F) = \mu(G)/[\mu(F) + \mu(G)]$. Then, the percentage error in truncating after N passes through the loop can be expressed as

$$\%E_N = \frac{E_N}{P_3(T)} 100\% = \rho(F)^{N+1} 100\% = \left[\frac{\mu(G)}{\mu(F)+\mu(G)}\right]^{N+1} 100\%$$

For $\mu(G) > \mu(F)$, many iterations of the loop, that is, large N, may be needed in order to reduce $\%E_N$ to a desirable level. If N^* is defined to be the minimum number of iterations through the loop needed to assure an $x\%$ error in the bounds, then

$$N^* = \frac{\ln(x\%/100)}{\ln\{\mu(G)/[\mu(F)+\mu(G)]\}}$$

Table 2 shows the approximate number of iterations through a loop as shown in figure 9 needed to guarantee a given percentage error in SURE's bounds. As demonstrated in this table, only a small number of iterations through the loop in the intermittent model are required to assure a small relative error in the bounds when $\mu(F)$ is larger than $\mu(G)$. A 5-percent error in the bounds where $\mu(F) \approx \varepsilon\mu(G)$ and ε approaches 0 requires $N^* \approx [-\ln(x\%/100)]/\varepsilon$ to make certain that the upper bound has adequately converged. In these cases, the user should try several values of the truncation parameter to make certain that the bounds have sufficiently converged.

Table 2. Sensitivity of SURE's Bounds to the Transition Probabilities of the General Transitions in an Intermittent-Fault Model

Ratio of the means of the general transitions F and G	Minimum number of iterations to guarantee $x\%$ error in SURE's bounds		
	$x = 1\%$	$x = 5\%$	$x = 10\%$
$\mu(F) \approx 100\mu(G)$	1	1	1
$\mu(F) \approx 10\mu(G)$	2	2	1
$\mu(F) \approx 5\mu(G)$	3	2	2
$\mu(F) \approx 3\mu(G)$	4	3	2
$\mu(F) \approx 2\mu(G)$	5	3	3
$\mu(F) \approx \mu(G)$	7	5	4
$\mu(F) \approx \mu(G)/2$	12	8	6
$\mu(F) \approx \mu(G)/4$	21	13	11
$\mu(F) \approx \mu(G)/10$	49	32	25
$\mu(F) \approx \mu(G)/100$	463	302	232
$\mu(F) \approx \mu(G)/1000$	4608	2998	2304
$\mu(F) \approx \mu(G)/10000$	46055	29959	23028

Overall, the relative error in the bounds is generally small. The relative error is particularly small when the parameters are in the range of those for highly reliable, fault-tolerant systems—that is, the fault-arrival rates are slow and the recovery rates are fast.

For transient faults, the bounds tend to converge quickly, so only small values for the truncation parameter are usually needed. For intermittent faults, the user should make judicious use of both the truncation and the pruning features to ensure that the bounds have adequately converged. The user should always be conscious of the sum of the probabilities of being in the pruned states to make sure that this quantity is not larger than the unreliability given in the bounds.

Concluding Remarks

Throughout this investigation of SURE's bounds, no cases were found where SURE's bounds did not envelop the exact unreliability or the best estimate of unreliability for any model tested. In general, the upper bound provided a very good estimate of the unreliability. For a few cases, the bounds were substantially separated; these cases had either fast fault-arrival rates or slow fault-recovery rates not typical of fault-tolerant computer systems. With judicious use of the truncation and pruning features, SURE produced good bounds for models with transient and intermittent faults. Overall, the SURE program provides a very good estimate in the form of lower and upper bounds of the unreliability of a semi-Markov model of a system.

In general, the program is very easy to use. SURE's input language is simple and direct and does not require the user to define parameters that are ambiguous or unmeasurable. The warning and error messages output by the program were helpful in determining the source of modeling and syntax problems. However, when truncation and pruning were used simultaneously, more attention should to be drawn to the sum of the probabilities of being in the pruned states. No cases were encountered where erroneous inputs were given which the program failed to catch and warn the user. The output is a concise and comprehensible presentation of the desired death-state probabilities of the model along with an error estimate.

The results of this study should give users of SURE confidence in the accuracy of SURE's bounds. Although many different models and test cases were used in this study, only the basic features of the program were analyzed. Testing every feature of the program is needed to build more confidence in SURE. All the features that were not considered in this study, such as the FAST, AUTOFAST, and ORPROB commands (ref. 3), need to be exercised over a range of models and parameter values. Beyond stress testing each of SURE's features, the program also needs to be tested to see whether a user could input an erroneous model without getting any warnings, or, in general, if any unacceptable input could be entered without the program adequately warning the user.

NASA Langley Research Center
Hampton, Virginia 23665-5225
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Appendix A

Comparison of SURE With Analytic Solutions

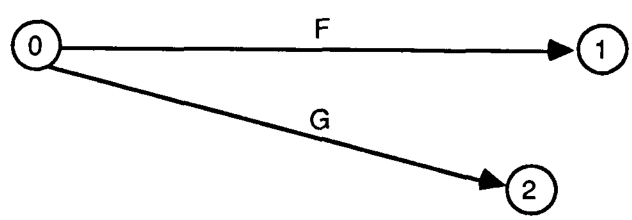
Example 1

The first example is a three-state semi-Markov model representing two competing uniform transitions. The distributions F and G are defined as follows:

$$F(t) = \begin{cases} t/a & (t \le a) \\ 1 & (t > a) \end{cases}$$

$$G(t) = \begin{cases} t/b & (t \le b) \\ 1 & (t > b) \end{cases}$$

where $a \le b \le T$. This first example is described in terms of the path-step classifications to help illustrate the connection between the equations given in the examples and the mathematics in the main body of the paper. In this example, both state 1 and state 2 are death states with a single path step leading to each state. The path from states 0 to 1 consists of a fast on-path transition with a fast off-path transition; hence, this is a class 2 path step. Similarly, the path from states 0 to 2 is also a class 2 path step.



Example 1: Three-state model with competing uniform transitions.

The following equations define the statistics needed to describe the general transitions F and G as class 2 paths for example 1:

$$\rho(F^*) = \int_0^\infty [1 - G(t)]\, dF(t) = \int_0^a \left(1 - \frac{t}{b}\right) \frac{1}{a}\, dt = 1 - \frac{a}{2b}$$

$$\mu(F^*) = \frac{1}{\rho(F^*)} \int_0^\infty t[1 - G(t)]\, dF(t) = \frac{2b}{2b - a} \int_0^a t \left(1 - \frac{t}{b}\right) \frac{1}{a}\, dt = \frac{3ab - 2a^2}{6b - 3a}$$

$$\sigma^2(F^*) = \frac{1}{\rho(F^*)} \int_0^\infty t^2[1 - G(t)]\, dF(t) - \mu^2(F^*) = \frac{2b}{2b - a} \int_0^a t^2 \left(1 - \frac{t}{b}\right) \frac{1}{a}\, dt - \mu^2(F^*)$$

$$= \frac{6a^2b^2 - 6a^3b + a^4}{72b^2 - 72ab + 18a^2}$$

$$\rho(G^*) = \int_0^\infty [1 - F(t)]\, dG(t) = \int_0^a \frac{1 - t/a}{b}\, dt = \frac{a}{2b}$$

$$\mu(G^*) = \frac{1}{\rho(G^*)} \int_0^\infty t[1 - F(t)]\, dG(t) = \frac{2b}{a} \int_0^a \frac{t}{b} \left(1 - \frac{t}{a}\right) dt = \frac{a}{3}$$

$$\sigma^2(G^*) = \frac{1}{\rho(G^*)} \int_0^\infty t^2[1 - F(t)]\, dG(t) - \mu^2(G^*) = \frac{2b}{a} \int_0^a \frac{t^2}{b} \left(1 - \frac{t}{a}\right) dt - \mu^2(G^*) = \frac{a^2}{18}$$

The following equations define the death-state probabilities for states 1 and 2 in example 1. The equations for $D_1(T)$ and $D_2(T)$ relate to $P_2(T)$, the probability of being in a death state at the end of a path consisting entirely of class 2 path steps, which is defined in the main text.

$$D_1(T) = \int_0^T [1 - G(t)]\, dF(t) = \int_0^a \left(1 - \frac{t}{b}\right) \frac{1}{a}\, dt = 1 - \frac{a}{2b}$$

$$D_2(T) = \int_0^T [1 - F(t)]\, dG(t) = \int_0^a \left(1 - \frac{t}{a}\right) \frac{1}{b}\, dt = \frac{a}{2b}$$

Comparison of SURE With Analytic Solutions for Example 1				
Parameters	Death states	Analytic solutions	SURE bounds	RD
$a = 1 \times 10^{-6}$	$D_1(T)$	9.50000×10^{-1}	$(9.49991 \times 10^{-1},\ 9.50000 \times 10^{-1})$	0.001
$b = 1 \times 10^{-5}$	$D_2(T)$	5.00000×10^{-2}	$(4.99996 \times 10^{-2},\ 5.00000 \times 10^{-2})$	.001
$a = 1 \times 10^{-6}$	$D_1(T)$	9.99995×10^{-1}	$(9.99986 \times 10^{-1},\ 9.99995 \times 10^{-1})$	.001
$b = 1 \times 10^{-1}$	$D_2(T)$	5.00000×10^{-6}	$(4.99996 \times 10^{-6},\ 5.00000 \times 10^{-6})$	.001
$a = 5 \times 10^{-8}$	$D_1(T)$	1.00000	$(9.99999 \times 10^{-1},\ 1.00000)$	.000
$b = 4 \times 10^{-8}$	$D_2(T)$	2.50000×10^{-9}	$(2.50000 \times 10^{-9},\ 2.50000 \times 10^{-9})$	.000
$a = 1 \times 10^{-3}$	$D_1(T)$	9.50000×10^{-1}	$(9.49114 \times 10^{-1},\ 9.50000 \times 10^{-1})$	.093
$b = 1 \times 10^{-2}$	$D_2(T)$	5.00000×10^{-2}	$(4.99627 \times 10^{-2},\ 5.00000 \times 10^{-2})$	.075
$a = 2 \times 10^{-2}$	$D_1(T)$	9.00000×10^{-1}	$(8.93878 \times 10^{-1},\ 9.00000 \times 10^{-1})$	.680
$b = 1 \times 10^{-1}$	$D_2(T)$	1.00000×10^{-1}	$(9.94497 \times 10^{-2},\ 1.00000 \times 10^{-1})$	.550
$a = 1 \times 10^{-4}$	$D_1(T)$	1.00000	$(9.99797 \times 10^{-1},\ 1.00000)$	.020
$b = 1 \times 10^{2}$	$D_2(T)$	5.00000×10^{-7}	$(4.99920 \times 10^{-7},\ 5.00000 \times 10^{-7})$	.016
$a = 1 \times 10^{-1}$	$D_1(T)$	1.00000	$(9.79725 \times 10^{-1},\ 1.00000)$	2.028
$b = 1 \times 10^{5}$	$D_2(T)$	5.00000×10^{-7}	$(4.91954 \times 10^{-7},\ 5.00000 \times 10^{-7})$	1.609

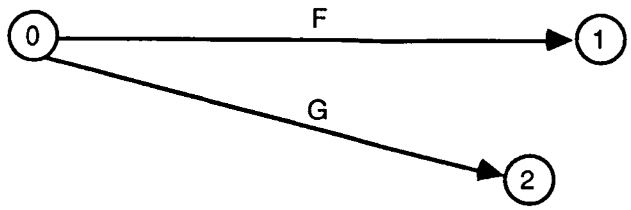
Example 2

The second example is a three-state semi-Markov model representing an impulse distribution competing with a uniform transition. The distributions F and G are defined as follows:

$$F(t) = \begin{cases} 0 & (t \leq a) \\ 1 & (t > a) \end{cases}$$

$$G(t) = \begin{cases} t/b & (t \leq b) \\ 1 & (t > b) \end{cases}$$

where $a \leq b \leq T$.



Example 2: Three-state model with impulse and uniform transitions.

The following equations define the statistics needed to describe the general transitions F and G for example 2:

$$\rho(F^*) = \int_0^\infty [1 - G(t)]\, dF(t) = 1 - G(a) = 1 - \frac{a}{b}$$

$$\mu(F^*) = \frac{1}{\rho(F^*)} \int_0^\infty t[1 - G(t)]\, dF(t) = \frac{b}{b-a} a \left(1 - \frac{a}{b}\right) = a$$

$$\sigma^2(F^*) = \frac{1}{\rho(F^*)} \int_0^\infty t^2[1 - G(t)]\, dF(t) - \mu^2(F^*) = \frac{b}{b-a} a^2 \left(1 - \frac{a}{b}\right) - \mu^2(F^*) = 0$$

$$\rho(G^*) = \int_0^\infty [1 - F(t)]\, dG(t) = \int_0^a \frac{1}{b}\, dt = \frac{a}{b}$$

$$\mu(G^*) = \frac{1}{\rho(G^*)} \int_0^\infty t[1 - F(t)]\, dG(t) = \frac{b}{a} \int_0^a \frac{t}{b}\, dt = \frac{a}{2}$$

$$\sigma^2(G^*) = \frac{1}{\rho(G^*)} \int_0^\infty t^2[1 - F(t)]\, dG(t) - \mu^2(G^*) = \frac{b}{a} \int_0^a \frac{t^2}{b}\, dt - \mu^2(G^*) = \frac{a^2}{12}$$

The following equations define the death-state probabilities for states 1 and 2 in example 2:

$$D_1(T) = \int_0^T [1 - G(t)]\, dF(t) = 1 - G(a) = 1 - \frac{a}{b}$$

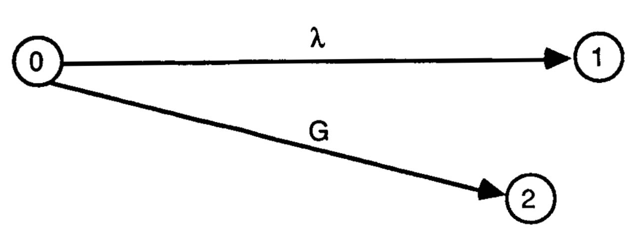
$$D_2(T) = \int_0^T [1 - F(t)]\, dG(t) = \int_0^a \frac{1}{b}\, dt = \frac{a}{b}$$

Comparison of SURE With Analytic Solutions for Example 2				
Parameters	Death states	Analytic solutions	SURE bounds	RD
$a = 1 \times 10^{-6}$ $b = 1 \times 10^{-5}$	$D_1(T)$ $D_2(T)$	9.00000×10^{-1} 1.00000×10^{-1}	$(8.99998 \times 10^{-1},\ 9.00000 \times 10^{-1})$ $(9.99991 \times 10^{-2},\ 1.00000 \times 10^{-1})$	0.001 .001
$a = 1 \times 10^{-6}$ $b = 1 \times 10^{-1}$	$D_1(T)$ $D_2(T)$	9.99990×10^{-1} 1.00000×10^{-5}	$(9.99976 \times 10^{-1},\ 9.99990 \times 10^{-1})$ $(9.99991 \times 10^{-6},\ 1.00000 \times 10^{-5})$	.001 .001
$a = 1 \times 10^{-2}$ $b = 1 \times 10^{-1}$	$D_1(T)$ $D_2(T)$	9.00000×10^{-1} 1.00000×10^{-1}	$(8.94330 \times 10^{-1},\ 9.00000 \times 10^{-1})$ $(9.95632 \times 10^{-2},\ 1.00000 \times 10^{-1})$	.630 .437
$a = 5 \times 10^{-7}$ $b = 2.5 \times 10^{-5}$	$D_1(T)$ $D_2(T)$	9.80000×10^{-1} 2.00000×10^{-2}	$(9.79992 \times 10^{-1},\ 9.80000 \times 10^{-1})$ $(1.99999 \times 10^{-2},\ 2.00000 \times 10^{-2})$	.001 .001
$a = 5 \times 10^{-7}$ $b = 1 \times 10^{-2}$	$D_1(T)$ $D_2(T)$	9.99950×10^{-1} 5.00000×10^{-5}	$(9.99941 \times 10^{-1},\ 9.99950 \times 10^{-1})$ $(4.99997 \times 10^{-5},\ 5.00000 \times 10^{-5})$	.001 .001
$a = 1 \times 10^{-4}$ $b = 1 \times 10^{3}$	$D_1(T)$ $D_2(T)$	1.00000 1.00000×10^{-7}	$(9.99708 \times 10^{-1},\ 1.00000)$ $(9.99797 \times 10^{-8},\ 1.00000 \times 10^{-7})$	.029 .020
$a = 1 \times 10^{-1}$ $b = 1 \times 10^{5}$	$D_1(T)$ $D_2(T)$	9.99999×10^{-1} 1.00000×10^{-6}	$(9.70759 \times 10^{-1},\ 9.99999 \times 10^{-1})$ $(9.79726 \times 10^{-7},\ 1.00000 \times 10^{-6})$	2.924 2.027

Example 3

The third example is a three-state semi-Markov model representing an exponential transition competing with an impulse function. The two distributions are defined as follows: the exponential distribution is given by $1 - e^{-\lambda t}$ for $t > 0$ and

$$G(t) = \begin{cases} 0 & (t < a) \\ 1 & (t \geq a) \end{cases}$$



Example 3: Three-state model with exponential and impulse transitions.

The following equations define the statistics needed to describe the general transition G in example 3:

$$\rho(G^*) = \int_0^\infty dG(t) = 1$$

$$\mu(G^*) = \frac{1}{\rho(G^*)} \int_0^\infty t\, dG(t) = a$$

$$\sigma^2(G^*) = \frac{1}{\rho(G^*)} \int_0^\infty t^2 dG(t) - \mu^2(G^*) = a^2 - a^2 = 0$$

The following equations define the death-state probabilities for states 1 and 2 in example 3:

$$D_1(T) = \int_0^T [1 - G(t)]\lambda e^{-\lambda t}\, dt = \int_0^a \lambda e^{-\lambda t}\, dt = 1 - e^{-\lambda a}$$

$$D_2(T) = \int_0^T e^{-\lambda t}\, dG(t) = 1 - (1 - e^{-\lambda a}) = e^{-\lambda a}$$

Comparison of SURE With Analytic Solutions for Example 3				
Parameters	Death states	Analytic solutions	SURE bounds	RD
$\lambda = 1 \times 10^{-5}$	$D_1(T)$	4.99988×10^{-5}	$(1.46434 \times 10^{-5},\ 5.00000 \times 10^{-5})$*	70.712
$a = 5$	$D_2(T)$	9.99950×10^{-1}	$(6.03100 \times 10^{-1},\ 1.00000)$	39.687
$\lambda = 1 \times 10^{-5}$	$D_1(T)$	1.00000×10^{-9}	$(9.96838 \times 10^{-10},\ 1.00000 \times 10^{-9})$	.316
$a = 1 \times 10^{-4}$	$D_2(T)$	1.00000	$(9.99708 \times 10^{-1},\ 1.00000)$	.029
$\lambda = 1 \times 10^{-2}$	$D_1(T)$	1.00000×10^{-10}	$(9.99968 \times 10^{-11},\ 1.00000 \times 10^{-10})$	.003
$a = 1 \times 10^{-8}$	$D_2(T)$	1.00000	$(9.99999 \times 10^{-1},\ 1.00000)$	.000
$\lambda = 1 \times 10^{-1}$	$D_1(T)$	1.98013×10^{-2}	$(1.69716 \times 10^{-2},\ 2.00000 \times 10^{-2})$	14.290
$a = 2 \times 10^{-1}$	$D_2(T)$	9.80199×10^{-1}	$(9.33584 \times 10^{-1},\ 1.00000)$	4.756
$\lambda = 2 \times 10^{-2}$	$D_1(T)$	1.99980×10^{-4}	$(1.93655 \times 10^{-4},\ 2.00000 \times 10^{-4})$	3.163
$a = 1 \times 10^{-2}$	$D_2(T)$	9.99800×10^{-1}	$(9.93500 \times 10^{-1},\ 1.00000)$	.630
$\lambda = 1 \times 10^{-4}$	$D_1(T)$	3.00000×10^{-10}	$(2.99836 \times 10^{-10},\ 3.00000 \times 10^{-10})$	.055
$a = 3 \times 10^{-6}$	$D_2(T)$	1.00000	$(9.99972 \times 10^{-1},\ 1.00000)$	.003
$\lambda = 1 \times 10^{-1}$	$D_1(T)$	1.00000×10^{-8}	$(9.99900 \times 10^{-9},\ 1.00000 \times 10^{-8})$	.010
$a = 1 \times 10^{-7}$	$D_2(T)$	1.00000	$(9.99997 \times 10^{-1},\ 1.00000)$	.000
$\lambda = 2 \times 10^{-3}$	$D_1(T)$	9.99500×10^{-4}	$(7.75893 \times 10^{-4},\ 1.00000 \times 10^{-3})$	22.372
$a = 5 \times 10^{-1}$	$D_2(T)$	9.99000×10^{-1}	$(9.13501 \times 10^{-1},\ 1.00000)$	8.558
$\lambda = 3 \times 10^{-4}$	$D_1(T)$	2.99955×10^{-4}	$(2.05087 \times 10^{-4},\ 3.00000 \times 10^{-4})$	31.627
$a = 1$	$D_2(T)$	9.99700×10^{-1}	$(8.63979 \times 10^{-1},\ 1.00000)$	13.576
$\lambda = 4 \times 10^{-7}$	$D_1(T)$	4.00000×10^{-10}	$(3.96000 \times 10^{-10},\ 4.00000 \times 10^{-10})$	1.000
$a = 1 \times 10^{-3}$	$D_2(T)$	1.00000	$(9.98643 \times 10^{-1},\ 1.00000)$	.136
$\lambda = 1 \times 10^{-5}$	$D_1(T)$	1.99998×10^{-5}	$(1.10555 \times 10^{-5},\ 2.00000 \times 10^{-5})$	44.722
$a = 2$	$D_2(T)$	9.99980×10^{-1}	$(7.84537 \times 10^{-1},\ 1.00000)$	21.545
$\lambda = 1 \times 10^{-5}$	$D_1(T)$	2.99996×10^{-5}	$(1.35679 \times 10^{-5},\ 3.00000 \times 10^{-5})$*	54.773
$a = 3$	$D_2(T)$	9.99970×10^{-1}	$(7.17659 \times 10^{-1},\ 1.00000)$	28.232
*RECOVERY TOO SLOW				

Note in the table that for values of $a > 0.1$, the relative difference is large. The SURE program has difficulty handling general recovery transitions which are slow relative to the mission time.

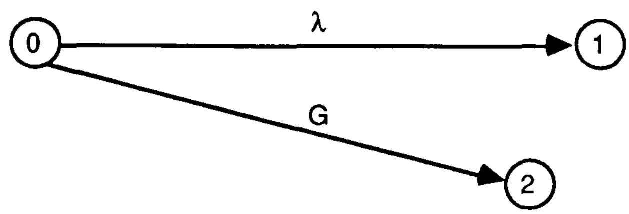
Example 4

Example 4a demonstrates the effect of using a slow exponential transition description when the exponential rate is actually fast. In contrast, example 4b shows the effect of using means and standard deviations to describe a slow exponential transition. Example 4b uses the same model as in 4a except the transition between states 0 and 1 is expressed as a general transition with mean and standard deviation. To demonstrate the problems associated with improper specification of an exponential transition, the same test cases containing a wide range of values for the exponential transition are given in the tables for examples 4a and 4b.

The model for these two examples is a three-state semi-Markov model representing an exponential transition competing with a uniform transition. The exponential distribution is given by $1 - e^{\lambda t}$ for $t > 0$ and

$$G(t) = \begin{cases} t/b & (t \leq b) \\ 1 & (t > b) \end{cases}$$

where $b < T$.



Example 4a: Three-state model with slow exponential and uniform transitions.

The following equations define the statistics needed to describe the general transition G in example 4a:

$$\rho(G^*) = \int_0^\infty dG(t) = 1$$

$$\mu(G^*) = \frac{1}{\rho(G^*)} \int_0^\infty t\, dG(t) = \int_0^b \frac{t}{b}\, dt = \frac{b}{2}$$

$$\sigma^2(G^*) = \frac{1}{\rho(G^*)} \int_0^\infty t^2\, dG(t) - \mu^2(G^*) = \int_0^b \frac{t^2}{b}\, dt - \frac{b^2}{4} = \frac{b^2}{12}$$

The following equations define the death-state probabilities for states 1 and 2 in example 4a:

$$D_1(T) = \int_0^T [1 - G(t)]\lambda e^{-\lambda t}\, dt = \int_0^b \left(1 - \frac{t}{b}\right) \lambda e^{-\lambda t} dt = \frac{\lambda b + e^{-\lambda b} - 1}{\lambda b}$$

$$D_2(T) = \int_0^T e^{-\lambda t} dG(t) = \int_0^b \frac{e^{-\lambda t}}{b}\, dt = \frac{1 - e^{-\lambda b}}{\lambda b}$$

Comparison of SURE With Analytic Solutions for Example 4a				
Parameters	Death states	Analytic solutions	SURE bounds	RD
$\lambda = 1 \times 10^{-6}$	$D_1(T)$	5.00000×10^{-11}	$(4.98709 \times 10^{-11},\ 5.00000 \times 10^{-11})$	0.258
$b = 1 \times 10^{-4}$	$D_2(T)$	1.00000	$(9.99797 \times 10^{-1},\ 1.00000)$	.203
$\lambda = 1 \times 10^{-4}$	$D_1(T)$	5.00000×10^{-9}	$(4.98709 \times 10^{-9},\ 5.00000 \times 10^{-9})$	.258
$b = 1 \times 10^{-4}$	$D_2(T)$	1.00000	$(9.99797 \times 10^{-1},\ 1.00000)$	.203
$\lambda = 1 \times 10^{-5}$	$D_1(T)$	5.00000×10^{-8}	$(4.87090 \times 10^{-8},\ 5.00000 \times 10^{-8})$	2.582
$b = 1 \times 10^{-2}$	$D_2(T)$	1.00000	$(9.95632 \times 10^{-1},\ 1.00000)$	.437
$\lambda = 1 \times 10^{-7}$	$D_1(T)$	5.00000×10^{-11}	$(4.95918 \times 10^{-11},\ 5.00000 \times 10^{-11})$	.816
$b = 1 \times 10^{-3}$	$D_2(T)$	1.00000	$(9.99059 \times 10^{-1},\ 1.00000)$	.094
$\lambda = 1 \times 10^{-2}$	$D_1(T)$	5.00000×10^{-7}	$(4.98709 \times 10^{-7},\ 5.00000 \times 10^{-7})$	.258
$b = 1 \times 10^{-4}$	$D_2(T)$	1.00000	$(9.99797 \times 10^{-1},\ 1.00000)$	.203
$\lambda = 1 \times 10^{2}$	$D_1(T)$	9.00005×10^{-2}	(0.00000, 1.00000)*	100.000
$b = 1 \times 10^{-1}$	$D_2(T)$	9.99955×10^{-1}	(0.00000, 1.00000)	100.000
$\lambda = 1 \times 10^{-4}$	$D_1(T)$	4.99998×10^{-6}	$(4.59173 \times 10^{-6},\ 5.00000 \times 10^{-6})$	8.165
$b = 1 \times 10^{-1}$	$D_2(T)$	9.99995×10^{-1}	$(9.79721 \times 10^{-1},\ 1.00000)$	2.027
$\lambda = 1 \times 10^{-1}$	$D_1(T)$	9.90000×10^{-1}	(0.00000, 1.00000)* † ‡	100.000
$b = 1 \times 10^{3}$	$D_2(T)$	1.00000×10^{-2}	(0.00000, 1.00000)	100.000
$\lambda = 1 \times 10^{2}$	$D_1(T)$	4.98337×10^{-3}	$(4.97042 \times 10^{-3},\ 5.00000 \times 10^{-3})$	.260
$b = 1 \times 10^{-4}$	$D_2(T)$	9.95017×10^{-1}	$(9.94797 \times 10^{-1},\ 1.00000)$	.022
$\lambda = 1 \times 10^{3}$	$D_1(T)$	9.99000×10^{-1}	(0.00000, 1.00000)*	100.000
$b = 1$	$D_2(T)$	1.00000×10^{-3}	(0.00000, 1.00000)	100.000
$\lambda = 1 \times 10^{5}$	$D_1(T)$	9.90000×10^{-1}	(0.00000, 1.00000)*	100.000
$b = 1 \times 10^{-3}$	$D_2(T)$	1.00000×10^{-2}	(0.00000, 1.00000)	100.000
$\lambda = 1 \times 10^{1}$	$D_1(T)$	4.99983×10^{-5}	$(4.99575 \times 10^{-5},\ 5.00000 \times 10^{-5})$	.082
$b = 1 \times 10^{-5}$	$D_2(T)$	9.99950×10^{-1}	$(9.99906 \times 10^{-1},\ 1.00000)$	.004
$\lambda = 1 \times 10^{4}$	$D_1(T)$	1.00000	(0.00000, 1.00000)* † ‡	100.000
$b = 1 \times 10^{3}$	$D_2(T)$	1.00000×10^{-7}	(0.00000, 1.00000)	100.000
$\lambda = 1 \times 10^{-5}$	$D_1(T)$	4.99983×10^{-5}	$(9.17350 \times 10^{-6},\ 5.00000 \times 10^{-5})$†	81.652
$b = 1 \times 10^{1}$	$D_2(T)$	9.99950×10^{-1}	$(5.63160 \times 10^{-1},\ 1.00000)$	43.681

*RATE TOO FAST
†RECOVERY TOO SLOW
‡DELTA > TIME

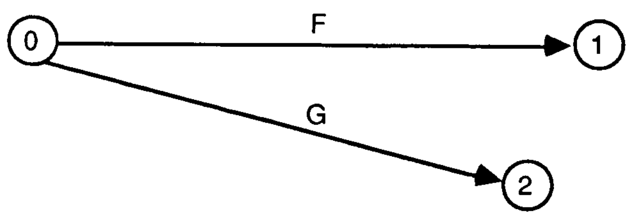
For large values of λ (i.e., a fast exponential transition) in the table for example 4a, the bounds separate except in cases where the competing recovery rate is very fast. These fast exponential rates should be expressed as general transitions with means and standard deviations.

The model for example 4b is the same as for example 4a except that the transition from state 0 to state 1 is now considered a general transition. The distributions F and G are defined as follows:

$$F(t) = 1 - e^{-\lambda t} \qquad (t > 0)$$

$$G(t) = \begin{cases} t/b & (t \le b) \\ 1 & (t > b) \end{cases}$$

As before, $b < T$.



Example 4b: Three-state model with fast exponential and uniform transitions.

The following equations define the statistics needed to describe the general transitions F and G for example 4b:

$$\rho(F^*) = \int_0^\infty [1 - G(t)]\, dF(t) = \int_0^b \left(1 - \frac{t}{b}\right) \lambda e^{-\lambda t}\, dt = \frac{\lambda b - 1 + e^{-\lambda b}}{\lambda b}$$

$$\mu(F^*) = \frac{1}{\rho(F^*)} \int_0^\infty t[1 - G(t)]\, dF(t) = \frac{1}{\rho(F^*)} \int_0^b t\left(1 - \frac{t}{b}\right) \lambda e^{-\lambda t}\, dt$$

$$= \frac{\lambda b - 2 + \lambda b e^{-\lambda b} + 2e^{-\lambda b}}{\lambda(\lambda b - 1 + e^{-\lambda b})}$$

$$\sigma^2(F^*) = \frac{1}{\rho(F^*)} \int_0^\infty t^2[1 - G(t)]\, dF(t) - \mu^2(F^*) = \frac{1}{\rho(F^*)} \int_0^b t^2\left(1 - \frac{t}{b}\right) \lambda e^{-\lambda t}\, dt$$

$$= \frac{2\lambda b - 6 + e^{-\lambda b}(\lambda^2 b^2 + 4\lambda b + 6)}{\lambda^2(\lambda b - 1 + e^{-\lambda b})} - \mu^2(F^*)$$

$$\rho(G^*) = \int_0^\infty [1 - F(t)]\, dG(t) = \int_0^\infty \frac{1}{b} e^{-\lambda t} dt = \frac{1 - e^{-\lambda b}}{\lambda b}$$

$$\mu(G^*) = \frac{1}{\rho(G^*)} \int_0^\infty t[1 - F(t)]\, dG(t) = \frac{1}{\rho(G^*)} \int_0^b \frac{t}{b} e^{-\lambda t}\, dt = \frac{1 - \lambda b e^{-\lambda b} - e^{-\lambda b}}{\lambda(1 - e^{-\lambda b})}$$

$$\sigma^2(G^*) = \frac{1}{\rho(G^*)} \int_0^\infty t^2[1 - F(t)]\, dG(t) - \mu^2(G^*) = \frac{1}{\rho(G^*)} \int_0^b \frac{t^2}{b} e^{-\lambda t}\, dt - \mu^2(G^*)$$

$$= \frac{2 - e^{-\lambda b}(\lambda^2 b^2 + 2\lambda b + 2)}{\lambda^2(1 - e^{-\lambda b})} - \mu^2(G^*)$$

The following equations define the death-state probabilities for states 1 and 2 in example 4b:

$$D_1(T) = \int_0^T [1 - G(t)]\, dF(t) = \int_0^b \left(1 - \frac{t}{b}\right) \lambda e^{-\lambda t}\, dt = \frac{\lambda b + e^{-\lambda b} - 1}{\lambda b}$$

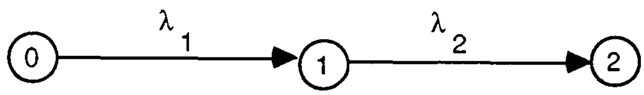
$$D_2(T) = \int_0^T [1 - F(t)]\, dG(t) = \int_0^b \frac{e^{-\lambda t}}{b} dt = \frac{1 - e^{-\lambda b}}{\lambda b}$$

Comparison of SURE With Analytic Solutions for Example 4b				
Parameters	Death states	Analytic solutions	SURE bounds	RD
$\lambda = 1 \times 10^{-6}$ $b = 1 \times 10^{-4}$	$D_1(T)$ $D_2(T)$	5.00000×10^{-11} 1.00000	$(4.99930 \times 10^{-11},\ 5.00000 \times 10^{-11})$ $(9.99816 \times 10^{-1},\ 1.00000)$	0.014 .018
$\lambda = 1 \times 10^{-4}$ $b = 1 \times 10^{-4}$	$D_1(T)$ $D_2(T)$	5.00000×10^{-9} 1.00000	$(4.99930 \times 10^{-9},\ 5.00000 \times 10^{-9})$ $(9.99816 \times 10^{-1},\ 1.00000)$	.014 .018
$\lambda = 1 \times 10^{-5}$ $b = 1 \times 10^{-2}$	$D_1(T)$ $D_2(T)$	5.00000×10^{-8} 1.00000	$(4.98421 \times 10^{-8},\ 5.00000 \times 10^{-8})$ $(9.96031 \times 10^{-1},\ 1.00000)$	.316 .397
$\lambda = 1 \times 10^{-7}$ $b = 1 \times 10^{-3}$	$D_1(T)$ $D_2(T)$	5.00000×10^{-11} 1.00000	$(4.99674 \times 10^{-11},\ 5.00000 \times 10^{-11})$ $(9.99145 \times 10^{-1},\ 1.00000)$	.065 .086
$\lambda = 1 \times 10^{-2}$ $b = 1 \times 10^{-4}$	$D_1(T)$ $D_2(T)$	5.00000×10^{-7} 1.00000	$(4.99930 \times 10^{-7},\ 5.00000 \times 10^{-7})$ $(9.99815 \times 10^{-1},\ 1.00000)$	.014 .019
$\lambda = 1 \times 10^{2}$ $b = 1 \times 10^{-1}$	$D_1(T)$ $D_2(T)$	9.00005×10^{-1} 9.99955×10^{-2}	$(8.94763 \times 10^{-1},\ 9.00005 \times 10^{-1})$ $(9.93657 \times 10^{-2},\ 9.99955 \times 10^{-2})$	.582 .630
$\lambda = 1 \times 10^{-4}$ $b = 1 \times 10^{-1}$	$D_1(T)$ $D_2(T)$	4.99998×10^{-6} 9.99995×10^{-1}	$(4.82881 \times 10^{-6},\ 4.99998 \times 10^{-6})$ $(9.81548 \times 10^{-1},\ 9.99995 \times 10^{-1})$	3.423 1.845
$\lambda = 1 \times 10^{-1}$ $b = 1 \times 10^{3}$	$D_1(T)$ $D_2(T)$	9.90000×10^{-1} 1.00000×10^{-2}	$(0.00000,\ 9.90000 \times 10^{-1})$*† $(0.00000,\ 1.00000 \times 10^{-2})$	100.000 100.000
$\lambda = 1 \times 10^{2}$ $b = 1 \times 10^{-4}$	$D_1(T)$ $D_2(T)$	4.98337×10^{-3} 9.95017×10^{-1}	$(4.98268 \times 10^{-3},\ 4.98337 \times 10^{-3})$ $(9.94834 \times 10^{-1},\ 9.95017 \times 10^{-1})$	.014 .018
$\lambda = 1 \times 10^{3}$ $b = 1$	$D_1(T)$ $D_2(T)$	9.99000×10^{-1} 1.00000×10^{-3}	$(9.97645 \times 10^{-1},\ 9.99000 \times 10^{-1})$ $(9.98643 \times 10^{-4},\ 1.00000 \times 10^{-3})$	.136 .136
$\lambda = 1 \times 10^{5}$ $b = 1 \times 10^{-3}$	$D_1(T)$ $D_2(T)$	9.90000×10^{-1} 1.00000×10^{-2}	$(9.89938 \times 10^{-1},\ 9.90000 \times 10^{-1})$ $(9.99937 \times 10^{-3},\ 1.00000 \times 10^{-2})$	.006 .006
$\lambda = 1 \times 10^{1}$ $b = 1 \times 10^{-5}$	$D_1(T)$ $D_2(T)$	4.99983×10^{-5} 9.99950×10^{-1}	$(4.99968 \times 10^{-5},\ 4.99983 \times 10^{-5})$ $(9.99910 \times 10^{-1},\ 9.99950 \times 10^{-1})$	.003 .004
$\lambda = 1 \times 10^{4}$ $b = 1 \times 10^{3}$	$D_1(T)$ $D_2(T)$	1.00000 1.00000×10^{-7}	$(9.99708 \times 10^{-1},\ 1.00000)$ $(9.99708 \times 10^{-8},\ 1.00000 \times 10^{-7})$	.029 .029
*RECOVERY TOO SLOW †DELTA > TIME				

The analytic solutions for the means and standard deviations were extremely numerically unstable for small values of λ and b. Consequently, a Taylor series expansion was used to reduce the form of the statistics used in the input files for the cases where λ and b were small.

Example 5

The fifth example is a three-state semi-Markov model with two different exponential transitions.



Example 5: Three-state model with slow exponential transitions.

The following equation defines the death-state probability for state 2 in example 5:

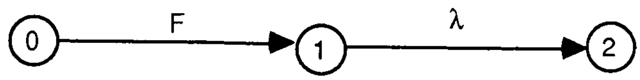
$$D_2(T) = \int_0^T \int_0^{T-x} \lambda_1 e^{-\lambda_1 x} \lambda_2 e^{-\lambda_2 y} dy dx = \frac{-\lambda_1 + \lambda_2 - \lambda_2 e^{-\lambda_1 T} + \lambda_1 e^{-\lambda_2 T}}{\lambda_2 - \lambda_1}$$

Comparison of SURE With Analytic Solutions for Example 5				
Parameters	Death states	Analytic solutions	SURE bounds	RD
$\lambda_1 = 1 \times 10^{-4}$ $\lambda_2 = 1 \times 10^{-3}$	$D_2(T)$	4.98171×10^{-6}	$(4.98167 \times 10^{-6},\ 5.00000 \times 10^{-6})$	0.367
$\lambda_1 = 1 \times 10^{-6}$ $\lambda_2 = 1 \times 10^{-1}$	$D_2(T)$	3.67878×10^{-6}	$(3.67878 \times 10^{-6},\ 3.67878 \times 10^{-6})$	.000
$\lambda_1 = 1 \times 10^{-2}$ $\lambda_2 = 1$	$D_2(T)$	8.60233×10^{-2}	$(8.60233 \times 10^{-2},\ 8.60233 \times 10^{-2})$	.000
$\lambda_1 = 1 \times 10^{-3}$ $\lambda_2 = 1 \times 10^{-5}$	$D_2(T)$	4.98321×10^{-7}	$(4.98317 \times 10^{-7},\ 5.00000 \times 10^{-7})$	.337
$\lambda_1 = 1$ $\lambda_2 = 1 \times 10^{-6}$	$D_2(T)$	9.00000×10^{-6}	$(9.00000 \times 10^{-6},\ 9.00000 \times 10^{-6})$	.000
$\lambda_1 = 1 \times 10^{-1}$ $\lambda_2 = 1 \times 10^{-7}$	$D_2(T)$	3.67879×10^{-7}	$(3.67879 \times 10^{-7},\ 3.67879 \times 10^{-7})$	.000
$\lambda_1 = 2 \times 10^{-5}$ $\lambda_2 = 3 \times 10^{-5}$	$D_2(T)$	2.99950×10^{-8}	$(2.99950 \times 10^{-8},\ 3.00000 \times 10^{-8})$	.017
$\lambda_1 = 1 \times 10^{-2}$ $\lambda_2 = 2 \times 10^{-2}$	$D_2(T)$	9.05592×10^{-3}	$(9.00000 \times 10^{-3},\ 1.00000 \times 10^{-2})$	10.425
$\lambda_1 = 8 \times 10^{-7}$ $\lambda_2 = 8.5 \times 10^{-7}$	$D_2(T)$	3.39999×10^{-11}	$(3.39998 \times 10^{-11},\ 3.40000 \times 10^{-11})$	.000
$\lambda_1 = 1 \times 10^{-7}$ $\lambda_2 = 2 \times 10^{-2}$	$D_2(T)$	9.36537×10^{-8}	$(9.33333 \times 10^{-8},\ 1.00000 \times 10^{-7})$	6.776

Example 6

In example 6, a three-state semi-Markov model with an impulse followed by an exponential transition is given. The impulse distribution F is defined as follows:

$$F(t) = \begin{cases} 0 & (t < a) \\ 1 & (t \geq a) \end{cases}$$



Example 6: Three-state model with impulse and exponential transitions.

The following equations define the statistics needed to describe the general transition F.

$$\rho(F^*) = \int_0^\infty dF(t) = 1$$

$$\mu(F^*) = \frac{1}{\rho(F^*)} \int_0^\infty t\,dF(t) = a$$

$$\sigma^2(F^*) = \frac{1}{\rho(F^*)} \int_0^\infty t^2\,dF(t) - \mu^2(F^*) = a^2 - a^2 = 0$$

The following equation defines the death-state probability for state 2 in example 6:

$$D_2(T) = \int_0^T \int_0^{T-x} \lambda e^{-\lambda y}\,dy\,dF(x) = 1 - e^{-\lambda(T-a)}$$

Comparison of SURE With Analytic Solutions for Example 6				
Parameters	Death states	Analytic solutions	SURE bounds	RD
$a = 1 \times 10^{-4}$ $\lambda = 1$	$D_2(T)$	9.99955×10^{-1}	$(9.99662 \times 10^{-1},\ 9.99955 \times 10^{-1})$	0.029
$a = 1$ $\lambda = 1 \times 10^{-2}$	$D_2(T)$	8.60688×10^{-2}	$(6.06740 \times 10^{-2},\ 1.00000 \times 10^{-1})$	29.505
$a = 1 \times 10^{-4}$ $\lambda = 1 \times 10^{-4}$	$D_2(T)$	9.99490×10^{-4}	$(9.98624 \times 10^{-4},\ 1.00000 \times 10^{-3})$	.087
$a = 1 \times 10^{-8}$ $\lambda = 1 \times 10^{-1}$	$D_2(T)$	6.32121×10^{-1}	$(6.32120 \times 10^{-1},\ 6.32121 \times 10^{-1})$	.002
$a = 1 \times 10^{-1}$ $\lambda = 1 \times 10^{-8}$	$D_2(T)$	9.90000×10^{-8}	$(9.13989 \times 10^{-8},\ 1.00000 \times 10^{-7})$	7.678
$a = 2 \times 10^{-5}$ $\lambda = 1 \times 10^{-6}$	$D_2(T)$	9.99993×10^{-6}	$(9.99695 \times 10^{-6},\ 1.00000 \times 10^{-5})$	.030
$a = 4 \times 10^{-5}$ $\lambda = 3 \times 10^{-2}$	$D_2(T)$	2.59181×10^{-1}	$(2.59070 \times 10^{-1},\ 2.59182 \times 10^{-1})$	.043
$a = 3 \times 10^{-7}$ $\lambda = 2 \times 10^{-4}$	$D_2(T)$	1.99800×10^{-3}	$(1.99796 \times 10^{-3},\ 2.00000 \times 10^{-3})$	.100
$a = 1 \times 10^{-6}$ $\lambda = 1 \times 10^{-5}$	$D_2(T)$	9.99950×10^{-5}	$(9.99909 \times 10^{-5},\ 1.00000 \times 10^{-4})$	.005
$a = 1.1 \times 10^{-3}$ $\lambda = 1 \times 10^{-3}$	$D_2(T)$	9.94908×10^{-3}	$(9.90702 \times 10^{-3},\ 1.00000 \times 10^{-2})$	.512
$a = 3 \times 10^{-4}$ $\lambda = 2 \times 10^{-3}$	$D_2(T)$	1.98007×10^{-2}	$(1.97641 \times 10^{-2},\ 2.00000 \times 10^{-2})$	1.007
$a = 2 \times 10^{-4}$ $\lambda = 1 \times 10^{-7}$	$D_2(T)$	9.99980×10^{-7}	$(9.98607 \times 10^{-7},\ 1.00000 \times 10^{-6})$	.137

Example 7

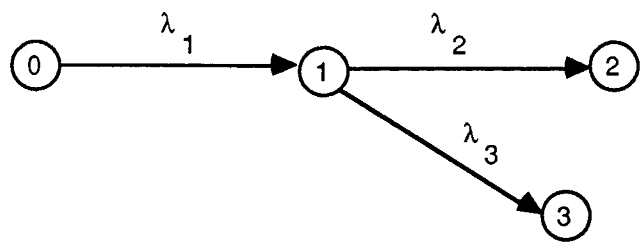
The seventh example is a four-state semi-Markov model with three exponential transitions. The following equations define the death-state probabilities for states 2 and 3 in example 7:

$$D_2(T) = \int_0^T \int_0^{T-x} \lambda_1 e^{-\lambda_1 x} \lambda_2 e^{-(\lambda_2+\lambda_3)y}\, dy\, dx$$

$$= \frac{\lambda_1 \lambda_3}{\lambda_2 + \lambda_3} \left[\frac{1}{\lambda_1} - \frac{e^{-\lambda_1 T}}{\lambda_1} - \frac{e^{-\lambda_1 T} - e^{-(\lambda_2+\lambda_3)T}}{\lambda_2 + \lambda_3 - \lambda_1} \right]$$

$$D_3(T) = \int_0^T \int_0^{T-x} \lambda_1 e^{-\lambda_1 x} \lambda_3 e^{-(\lambda_2+\lambda_3)y}\, dy\, dx$$

$$= \frac{\lambda_1\lambda_3}{\lambda_2+\lambda_3}\left[\frac{1}{\lambda_1} - \frac{e^{-\lambda_1 T}}{\lambda_1} - \frac{e^{-\lambda_1 T} - e^{-(\lambda_2+\lambda_3)T}}{\lambda_2+\lambda_3-\lambda_1}\right]$$

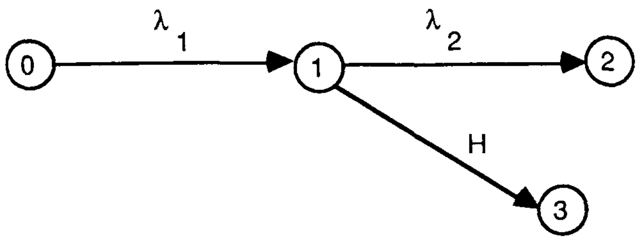


Example 7: Four-state model with exponential transitions.

Comparison of SURE With Analytic Solutions for Example 7				
Parameters	Death states	Analytic solutions	SURE bounds	RD
$\lambda_1 = 1 \times 10^{-2}$ $\lambda_2 = 1 \times 10^{-3}$ $\lambda_3 = 1 \times 10^{-4}$	$D_2(T)$ $D_3(T)$	4.81958×10^{-4} 4.81958×10^{-5}	$(4.81500 \times 10^{-4},\ 5.00000 \times 10^{-4})$ $(4.81500 \times 10^{-5},\ 5.00000 \times 10^{-5})$	3.743 3.743
$\lambda_1 = 1 \times 10^{-7}$ $\lambda_2 = 1 \times 10^{-2}$ $\lambda_3 = 1 \times 10^{-5}$	$D_2(T)$ $D_3(T)$	4.83726×10^{-8} 4.83726×10^{-11}	$(4.83316 \times 10^{-8},\ 5.00000 \times 10^{-8})$ $(4.83317 \times 10^{-11},\ 5.00000 \times 10^{-11})$	3.364 3.364
$\lambda_1 = 1 \times 10^{-5}$ $\lambda_2 = 2 \times 10^{-5}$ $\lambda_3 = 5 \times 10^{-5}$	$D_2(T)$ $D_3(T)$	9.99733×10^{-9} 2.49933×10^{-8}	$(9.99733 \times 10^{-9},\ 1.00000 \times 10^{-8})$ $(2.49933 \times 10^{-8},\ 2.50000 \times 10^{-8})$	.027 .027
$\lambda_1 = 1 \times 10^{-6}$ $\lambda_2 = 1 \times 10^{-7}$ $\lambda_3 = 1 \times 10^{-2}$	$D_2(T)$ $D_3(T)$	4.83740×10^{-12} 4.83740×10^{-7}	$(4.83332 \times 10^{-12},\ 5.00000 \times 10^{-12})$ $(4.83332 \times 10^{-7},\ 5.00000 \times 10^{-7})$	3.361 3.361
$\lambda_1 = 1 \times 10^{-1}$ $\lambda_2 = 1 \times 10^{-7}$ $\lambda_3 = 2 \times 10^{-7}$	$D_2(T)$ $D_3(T)$	3.67879×10^{-7} 7.35758×10^{-7}	$(3.67879 \times 10^{-7},\ 3.67879 \times 10^{-7})$ $(7.35758 \times 10^{-7},\ 7.35758 \times 10^{-7})$	.000 .000
$\lambda_1 = 1 \times 10^{-2}$ $\lambda_2 = 5 \times 10^{-5}$ $\lambda_3 = 3 \times 10^{-8}$	$D_2(T)$ $D_3(T)$	2.41830×10^{-5} 1.45098×10^{-8}	$(2.41625 \times 10^{-5},\ 2.50000 \times 10^{-5})$ $(1.44975 \times 10^{-8},\ 1.50000 \times 10^{-8})$	3.378 3.378

Example 8

The eighth example is a four-state semi-Markov model with two exponential transitions and one uniform transition. The uniform distribution H is defined as follows:



Example 8: Four-state model with exponential and uniform transitions.

$$H(t) = \begin{cases} t/b & (t \leq b) \\ 1 & (t > b) \end{cases}$$

where $b \leq T$.

The following equations define the statistics needed to describe the general transition H in example 8:

$$\rho(H^*) = \int_0^\infty dH(t) = 1$$

$$\mu(H^*) = \frac{1}{\rho(H^*)} \int_0^\infty t\, dH(t) = \frac{1}{b} \int_0^b t\, dt = \frac{b}{2}$$

$$\sigma^2(H^*) = \frac{1}{\rho(H^*)} \int_0^\infty t^2\, dH(t) - \mu^2(H^*) = \frac{1}{b} \int_0^b t^2\, dt - \frac{b^2}{4} = \frac{b^2}{12}$$

The following equations define the death-state probabilities for states 2 and 3 in example 8:

$$D_2(T) = \int_0^T \int_0^{T-x} \lambda_1 e^{-\lambda_1 x} \lambda_2 e^{-\lambda_2 y} [1 - H(y)]\, dy\, dx$$

$$= \lambda_1 \lambda_2 \int_0^T e^{-\lambda_1 x} \int_0^b e^{-\lambda_2 y} \left(1 - \frac{y}{b}\right) dy\, dx = 1 - \frac{1 - e^{-\lambda_2 b}}{\lambda_2 b} (1 - e^{-\lambda_1 T})$$

$$D_3(T) = \int_0^T \int_0^{T-x} \lambda_1 e^{-\lambda_1 x} e^{-\lambda_2 y}\, dH(y)\, dx = \frac{\lambda_1}{b} \int_0^T e^{-\lambda_1 x} \int_0^b e^{-\lambda_2 y}\, dy\, dx$$

$$= \frac{(1 - e^{-\lambda_2 b})(1 - e^{-\lambda_1 T})}{\lambda_2 b}$$

Comparison of SURE With Analytic Solutions for Example 8				
Parameters	Death states	Analytic solutions	SURE bounds	RD
$\lambda_1 = 4 \times 10^{-3}$				
$\lambda_2 = 3 \times 10^{-3}$	$D_2(T)$	5.88160×10^{-10}	$(5.87050 \times 10^{-10},\ 6.00000 \times 10^{-10})$	2.013
$b = 1 \times 10^{-5}$	$D_3(T)$	3.92107×10^{-2}	$(3.91949 \times 10^{-2},\ 4.00000 \times 10^{-2})$	2.013
$\lambda_1 = 5 \times 10^{-6}$				
$\lambda_2 = 3 \times 10^{-6}$	$D_2(T)$	7.49981×10^{-14}	$(7.37784 \times 10^{-14},\ 7.50000 \times 10^{-14})$	1.626
$b = 1 \times 10^{-3}$	$D_3(T)$	4.99987×10^{-5}	$(4.98577 \times 10^{-5},\ 5.00000 \times 10^{-5})$	.282
$\lambda_1 = 5 \times 10^{-4}$				
$\lambda_2 = 2 \times 10^{-3}$	$D_2(T)$	4.98752×10^{-10}	$(4.96181 \times 10^{-10},\ 5.00000 \times 10^{-10})$	.515
$b = 1 \times 10^{-4}$	$D_3(T)$	4.98752×10^{-3}	$(4.98447 \times 10^{-3},\ 5.00000 \times 10^{-3})$	.250
$\lambda_1 = 3 \times 10^{-2}$				
$\lambda_2 = 2 \times 10^{-2}$	$D_2(T)$	2.59182×10^{-9}	$(2.59057 \times 10^{-9},\ 2.59182 \times 10^{-9})$	.048
$b = 1 \times 10^{-6}$	$D_3(T)$	2.59182×10^{-1}	$(2.59175 \times 10^{-1},\ 2.59182 \times 10^{-1}$	.003
$\lambda_1 = 1 \times 10^{-2}$				
$\lambda_2 = 1 \times 10^{-5}$	$D_2(T)$	4.75813×10^{-10}	$(4.67476 \times 10^{-10},\ 5.00000 \times 10^{-10})$	5.083
$b = 1 \times 10^{-3}$	$D_3(T)$	9.51626×10^{-2}	$(9.47414 \times 10^{-2},\ 1.00000 \times 10^{-1})$	5.083
$\lambda_1 = 3 \times 10^{-6}$				
$\lambda_2 = 2 \times 10^{-6}$	$D_2(T)$	2.99996×10^{-17}	$(2.99841 \times 10^{-17},\ 3.00000 \times 10^{-17})$	.052
$b = 1 \times 10^{-6}$	$D_3(T)$	2.99996×10^{-5}	$(2.99987 \times 10^{-5},\ 3.00000 \times 10^{-5})$	.003
$\lambda_1 = 1 \times 10^{-3}$				
$\lambda_2 = 1 \times 10^{-2}$	$D_2(T)$	4.97508×10^{-12}	$(4.97419 \times 10^{-12},\ 5.00000 \times 10^{-12})$	.501
$b = 1 \times 10^{-7}$	$D_3(T)$	9.95017×10^{-3}	$(9.94994 \times 10^{-3},\ 1.00000 \times 10^{-2})$	.501
$\lambda_1 = 2 \times 10^{-5}$				
$\lambda_2 = 2 \times 10^{-5}$	$D_2(T)$	3.99960×10^{-14}	$(3.99037 \times 10^{-14},\ 4.00000 \times 10^{-14})$	.231
$b = 2 \times 10^{-5}$	$D_3(T)$	1.99980×10^{-4}	$(1.99938 \times 10^{-4},\ 2.00000 \times 10^{-4})$	.021
$\lambda_1 = 5 \times 10^{-5}$				
$\lambda_2 = 4 \times 10^{-5}$	$D_2(T)$	9.99749×10^{-10}	$(8.43172 \times 10^{-10},\ 1.00000 \times 10^{-10})$	15.662
$b = 1 \times 10^{-1}$	$D_3(T)$	4.99874×10^{-4}	$(4.69886 \times 10^{-4},\ 5.00000 \times 10^{-4})$	5.999
$\lambda_1 = 2 \times 10^{-3}$				
$\lambda_2 = 1 \times 10^{-6}$	$D_2(T)$	9.90066×10^{-10}	$(8.35622 \times 10^{-10},\ 1.00000 \times 10^{-9})$	15.599
$b = 1 \times 10^{-1}$	$D_3(T)$	1.98013×10^{-2}	$(1.86196 \times 10^{-2},\ 2.00000 \times 10^{-2})$	5.968
$\lambda_1 = 8 \times 10^{-2}$				
$\lambda_2 = 7 \times 10^{-2}$	$D_2(T)$	1.92730×10^{-5}	$(1.90135 \times 10^{-5},\ 1.92735 \times 10^{-5})$	1.346
$b = 1 \times 10^{-3}$	$D_3(T)$	5.50652×10^{-1}	$(5.49457 \times 10^{-1},\ 5.50671 \times 10^{-1})$	.217
$\lambda_1 = 4 \times 10^{-4}$				
$\lambda_2 = 3 \times 10^{-4}$	$D_2(T)$	5.98742×10^{-7}	$(3.29627 \times 10^{-7},\ 6.00000 \times 10^{-7})$	44.947
$b = 1$	$D_3(T)$	3.99141×10^{-3}	$(2.93634 \times 10^{-3},\ 4.00000 \times 10^{-3})$	26.434

In example 8, SURE's bounds separated in cases where the recovery rate was slow with respect to the mission time of 10 hours.

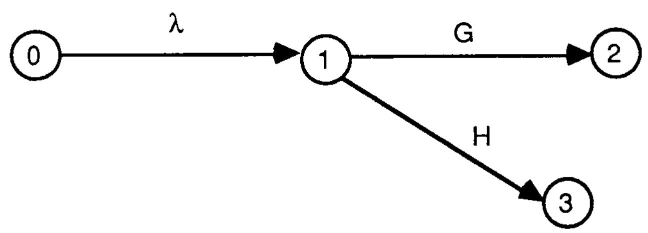
Example 9

The ninth example is a four-state semi-Markov model with an exponential transition, an impulse transition, and a uniform transition. The impulse distribution $G(t)$ and uniform distribution $H(t)$ are defined as follows:

$$G(t) = \begin{cases} 0 & (t < a) \\ 1 & (t \geq a) \end{cases}$$

$$H(t) = \begin{cases} t/b & (t \leq b) \\ 1 & (t > b) \end{cases}$$

where $a < b < T$.



Example 9: Four-state model with exponential, impulse, and uniform transitions.

The following equations define the statistics needed to describe the general transitions G and H in example 9:

$$\rho(G^*) = \int_0^\infty [1 - H(t)]\, dG(t) = 1 - H(a) = \frac{b-a}{b}$$

$$\mu(G^*) = \frac{1}{\rho(G^*)} \int_0^\infty t[1 - H(t)]\, dG(t) = \frac{b}{b-a} a[1 - H(a)] = a$$

$$\sigma^2(G^*) = \frac{1}{\rho(G^*)} \int_0^\infty t^2[1 - H(t)]\, dG(t) - \mu^2(G^*)$$

$$= \frac{b}{b-a} a^2[1 - H(a)] - a^2 = 0$$

$$\rho(H^*) = \int_0^\infty [1 - G(t)]\, dH(t) = \int_0^a \frac{1}{b}\, dt = \frac{a}{b}$$

$$\mu(H^*) = \frac{1}{\rho(H^*)} \int_0^\infty t[1 - G(t)]\, dH(t) = \frac{b}{a} \int_0^a \frac{t}{b}\, dt = \frac{a}{2}$$

$$\sigma^2(H^*) = \frac{1}{\rho(H^*)} \int_0^\infty t^2[1 - G(t)]\, dH(t) - \mu^2(H^*) = \frac{b}{a} \int_0^a \frac{t^2}{b}\, dt - \frac{a^2}{4} = \frac{a^2}{12}$$

The following equations define the death-state probabilities for states 2 and 3 in example 9:

$$
\begin{aligned}
D_2(T) &= \int_0^T \int_0^{T-x} \lambda e^{-\lambda x}[1 - H(y)]\, dG(y)\, dx \\
&= \lambda \int_0^{T-a} e^{-\lambda x} \left(1 - \frac{a}{b}\right)\, dx = \frac{(b-a)}{b}[1 - e^{-\lambda(T-a)}]
\end{aligned}
$$

$$
\begin{aligned}
D_3(T) &= \int_0^T \int_0^{T-x} \lambda e^{-\lambda x}[1 - G(y)]\, dH(y)\, dx \\
&= \int_0^{T-a} e^{-\lambda x} \int_0^a \frac{1}{b}\, dy\, dx + \lambda \int_{T-a}^{T} e^{-\lambda x} \int_0^{T-x} \frac{1}{b}\, dy\, dx \\
&= \frac{a}{b}[1 - e^{-\lambda(T-a)}] + \frac{T}{b}[e^{-\lambda(T-a)} - e^{-\lambda T}] \\
&\quad + \frac{1}{b}[Te^{-\lambda T} - (T-a)e^{-\lambda(T-a)}] - \frac{e^{-\lambda(T-a)} - e^{-\lambda T}}{\lambda b}
\end{aligned}
$$

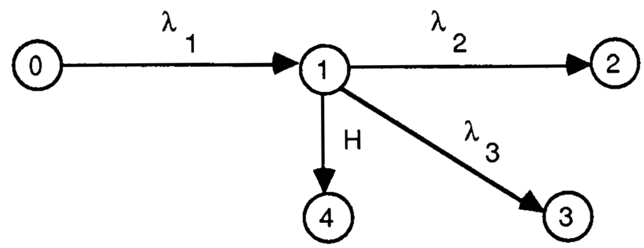
Comparison of SURE With Analytic Solutions for Example 9				
Parameters	Death states	Analytic solutions	SURE bounds	RD
$\lambda = 1 \times 10^{-2}$ $a = 1 \times 10^{-7}$ $b = 1 \times 10^{-3}$	 $D_2(T)$ $D_3(T)$	 9.51531×10^{-2} 9.51626×10^{-6}	 $(9.49897 \times 10^{-2},\ 9.99900 \times 10^{-2})$ $(9.49994 \times 10^{-6},\ 1.00000 \times 10^{-5})$	 5.083 5.083
$\lambda = 1 \times 10^{-3}$ $a = 1 \times 10^{-3}$ $b = 1 \times 10^{-2}$	 $D_2(T)$ $D_3(T)$	 8.95426×10^{-3} 9.94967×10^{-4}	 $(8.91869 \times 10^{-3},\ 9.00000 \times 10^{-3})$ $(9.92202 \times 10^{-4},\ 1.00000 \times 10^{-3})$	 .511 .506
$\lambda = 1 \times 10^{-6}$ $a = 1 \times 10^{-2}$ $b = 2 \times 10^{-2}$	 $D_2(T)$ $D_3(T)$	 4.99498×10^{-6} 4.99748×10^{-6}	 $(4.90588 \times 10^{-6},\ 5.00000 \times 10^{-6})$ $(4.93465 \times 10^{-6},\ 5.00000 \times 10^{-6})$	 1.784 1.257
$\lambda = 1 \times 10^{-1}$ $a = 1 \times 10^{-6}$ $b = 1 \times 10^{-5}$	 $D_2(T)$ $D_3(T)$	 5.68908×10^{-1} 6.32121×10^{-2}	 $(5.68892 \times 10^{-1},\ 5.68909 \times 10^{-1})$ $(6.32108 \times 10^{-2},\ 6.32121 \times 10^{-1})$	 .003 .002
$\lambda = 1 \times 10^{-2}$ $a = 1 \times 10^{-5}$ $b = 3 \times 10^{-5}$	 $D_2(T)$ $D_3(T)$	 6.34417×10^{-2} 3.17208×10^{-2}	 $(6.33218 \times 10^{-2},\ 6.66667 \times 10^{-2})$ $(3.16627 \times 10^{-2},\ 3.33333 \times 10^{-2})$	 5.083 5.083
$\lambda = 1 \times 10^{-4}$ $a = 1 \times 10^{-7}$ $b = 1 \times 10^{-1}$	 $D_2(T)$ $D_3(T)$	 9.99499×10^{-4} 9.99500×10^{-10}	 $(9.99490 \times 10^{-4},\ 9.99999 \times 10^{-4})$ $(9.99494 \times 10^{-10},\ 1.00000 \times 10^{-9})$	 .050 .050
$\lambda = 1 \times 10^{-4}$ $a = 1 \times 10^{-6}$ $b = 1 \times 10^{-5}$	 $D_2(T)$ $D_3(T)$	 8.99550×10^{-4} 9.99500×10^{-5}	 $(8.99513 \times 10^{-4},\ 9.00000 \times 10^{-4})$ $(9.99472 \times 10^{-5},\ 1.00000 \times 10^{-4})$	 .050 .050
$\lambda = 1 \times 10^{-1}$ $a = 1 \times 10^{-3}$ $b = 1 \times 10^{-2}$	 $D_2(T)$ $D_3(T)$	 5.68875×10^{-1} 6.32102×10^{-2}	 $(5.67238 \times 10^{-1},\ 5.68909 \times 10^{-1})$ $(6.30833 \times 10^{-2},\ 6.32121 \times 10^{-2})$	 .288 .201
$\lambda = 1 \times 10^{-6}$ $a = 1 \times 10^{-7}$ $b = 1 \times 10^{-6}$	 $D_2(T)$ $D_3(T)$	 8.99995×10^{-6} 9.99995×10^{-7}	 $(8.99988 \times 10^{-6},\ 9.00000 \times 10^{-6})$ $(9.99989 \times 10^{-7},\ 1.00000 \times 10^{-6})$	 .001 .001
$\lambda = 1 \times 10^{-3}$ $a = 1 \times 10^{-2}$ $b = 1 \times 10^{4}$	 $D_2(T)$ $D_3(T)$	 9.94026×10^{-3} 9.94522×10^{-9}	 $(9.76335 \times 10^{-3},\ 9.99999 \times 10^{-3})$ $(9.82043 \times 10^{-9},\ 1.00000 \times 10^{-8})$	 1.780 1.255
$\lambda = 1 \times 10^{-7}$ $a = 1$ $b = 1 \times 10^{6}$	 $D_2(T)$ $D_3(T)$	 8.99999×10^{-7} 9.50000×10^{-13}	 $(6.29677 \times 10^{-7},\ 9.99999 \times 10^{-7})$ $(7.35400 \times 10^{-13},\ 1.00000 \times 10^{-12})$	 30.036 22.589

Example 10

The tenth example is a five-state semi-Markov model with two exponential transitions competing with an impulse function. The impulse distribution $H(t)$ is defined as follows:

$$H(t) = \begin{cases} 0 & (t \leq a) \\ 1 & (t > a) \end{cases}$$

where $a < T$.



Example 10: Five-state model with exponential and impulse transitions.

The following equations define the statistics needed to describe the general transition H in example 10:

$$\rho(H^*) = \int_0^\infty dH(t) = 1$$

$$\mu(H^*) = \frac{1}{\rho(H^*)} \int_0^\infty t\, dH(t) = a$$

$$\sigma^2(H^*) = \frac{1}{\rho(H^*)} \int_0^\infty t^2\, dH(t) - \mu^2(H^*) = a^2 - a^2 = 0$$

The following equations define the death-state probabilities for states 2, 3, and 4 in example 10:

$$\begin{aligned} D_2(T) &= \int_0^T \int_0^{T-x} \lambda_1 e^{-\lambda_1 x} \lambda_2 e^{-(\lambda_2+\lambda_3)y} [1 - H(y)]\, dy\, dx \\ &= \lambda_1\lambda_2 \left[\int_0^{T-a} e^{-\lambda_1 x} \int_0^a e^{-(\lambda_2+\lambda_3)y}\, dy\, dx + \int_{T-a}^T e^{-\lambda_1 x} \int_0^{T-x} e^{-(\lambda_2+\lambda_3)y}\, dy\, dx \right] \\ &= \frac{\lambda_2[1 - e^{-(\lambda_2+\lambda_3)a}][1 - e^{-\lambda_1(T-a)}]}{\lambda_2 + \lambda_3} + \frac{\lambda_2[e^{-\lambda_1(T-a)} - e^{-\lambda_1 T}]}{\lambda_2 + \lambda_3} \\ &\quad - \frac{\lambda_1\lambda_2 e^{-(\lambda_2+\lambda_3)T}[e^{(-\lambda_1+\lambda_2+\lambda_3)T} - e^{(-\lambda_1+\lambda_2+\lambda_3)(T-a)}]}{(\lambda_2 + \lambda_3)(-\lambda_1 + \lambda_2 + \lambda_3)} \end{aligned}$$

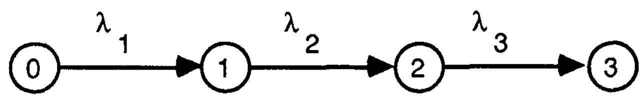
$$
\begin{aligned}
D_3(T) &= \int_0^T \int_0^{T-x} \lambda_1 e^{-\lambda_1 x} \lambda_3 e^{-(\lambda_2+\lambda_3)y}[1 - H(y)]\, dy\, dx \\
&= \lambda_1\lambda_3 \left[\int_0^{T-a} e^{-\lambda_1 x} \int_0^{a} e^{-(\lambda_2+\lambda_3)y}\, dy\, dx + \int_{T-a}^{T} e^{-\lambda_1 x} \int_0^{T-x} e^{-(\lambda_2+\lambda_3)y}\, dy\, dx\right] \\
&= \frac{\lambda_3[1 - e^{-(\lambda_2+\lambda_3)a}][1 - e^{-\lambda_1(T-a)}]}{\lambda_2+\lambda_3} + \frac{\lambda_3[e^{-\lambda_1(T-a)} - e^{-\lambda_1 T}]}{\lambda_2+\lambda_3} \\
&\quad - \frac{\lambda_1\lambda_3 e^{-(\lambda_2+\lambda_3)T}[e^{(-\lambda_1+\lambda_2+\lambda_3)T} - e^{(-\lambda_1+\lambda_2+\lambda_3)(T-a)}]}{(\lambda_2+\lambda_3)(-\lambda_1+\lambda_2+\lambda_3)}
\end{aligned}
$$

$$
\begin{aligned}
D_4(T) &= \int_0^T \int_0^{T-x} \lambda_1 e^{-\lambda_1 x} e^{-(\lambda_2+\lambda_3)y}\, dH(y)\, dx = \lambda_1 \int_0^{T-a} e^{-\lambda_1 x} e^{-(\lambda_2+\lambda_3)a}\, dx \\
&= e^{-(\lambda_2+\lambda_3)a}[1 - e^{-\lambda_1(T-a)}]
\end{aligned}
$$

Comparison of SURE With Analytic Solutions for Example 10				
Parameters	Death states	Analytic solutions	SURE bounds	RD
$\lambda_1 = 5 \times 10^{-4}$				
$\lambda_2 = 4 \times 10^{-4}$	$D_2(T)$	1.99501×10^{-12}	$(1.99374 \times 10^{-12},\ 2.00000 \times 10^{-12})$	0.250
$\lambda_3 = 3 \times 10^{-4}$	$D_3(T)$	1.49626×10^{-12}	$(1.49531 \times 10^{-12},\ 1.50000 \times 10^{-12})$	.250
$a = 1 \times 10^{-6}$	$D_4(T)$	4.98752×10^{-3}	$(4.98730 \times 10^{-3},\ 5.00000 \times 10^{-3})$	.250
$\lambda_1 = 1 \times 10^{-3}$				
$\lambda_2 = 1 \times 10^{-5}$	$D_2(T)$	9.95017×10^{-14}	$(9.94372 \times 10^{-14},\ 1.00000 \times 10^{-13})$	.501
$\lambda_3 = 1 \times 10^{-4}$	$D_3(T)$	9.95012×10^{-13}	$(9.94372 \times 10^{-13},\ 1.00000 \times 10^{-12})$	.501
$a = 1 \times 10^{-6}$	$D_4(T)$	9.95017×10^{-3}	$(9.94960 \times 10^{-3},\ 1.00000 \times 10^{-2})$	.501
$\lambda_1 = 3 \times 10^{-3}$				
$\lambda_2 = 2 \times 10^{-3}$	$D_2(T)$	1.18218×10^{-9}	$(1.17868 \times 10^{-9},\ 1.20000 \times 10^{-9})$	1.507
$\lambda_3 = 1 \times 10^{-5}$	$D_3(T)$	5.91089×10^{-12}	$(5.89342 \times 10^{-12},\ 6.00000 \times 10^{-12})$	1.508
$a = 2 \times 10^{-5}$	$D_4(T)$	2.95545×10^{-2}	$(2.95412 \times 10^{-2},\ 3.00000 \times 10^{-2})$	1.507
$\lambda_1 = 2 \times 10^{-2}$				
$\lambda_2 = 2 \times 10^{-3}$	$D_2(T)$	3.62495×10^{-5}	$(2.94801 \times 10^{-5},\ 4.00000 \times 10^{-5})$	18.674
$\lambda_3 = 4 \times 10^{-4}$	$D_3(T)$	7.24990×10^{-6}	$(5.89601 \times 10^{-6},\ 8.00000 \times 10^{-6})$	18.675
$a = 1 \times 10^{-1}$	$D_4(T)$	1.81226×10^{-1}	$(1.65546 \times 10^{-1},\ 2.00000 \times 10^{-1})$	10.359
$\lambda_1 = 5 \times 10^{-6}$				
$\lambda_2 = 4 \times 10^{-6}$	$D_2(T)$	1.99994×10^{-13}	$(1.96014 \times 10^{-13},\ 2.00000 \times 10^{-13})$	1.990
$\lambda_3 = 1 \times 10^{-2}$	$D_3(T)$	4.99985×10^{-10}	$(4.90035 \times 10^{-10},\ 5.00000 \times 10^{-10})$	1.990
$a = 1 \times 10^{-3}$	$D_4(T)$	4.99983×10^{-5}	$(4.97949 \times 10^{-5},\ 5.00000 \times 10^{-5})$	.407
$\lambda_1 = 1 \times 10^{-1}$				
$\lambda_2 = 1 \times 10^{-1}$	$D_2(T)$	3.16060×10^{-8}	$(3.15948 \times 10^{-8},\ 3.16060 \times 10^{-8})$	.035
$\lambda_3 = 2 \times 10^{-5}$	$D_3(T)$	6.32121×10^{-12}	$(6.31897 \times 10^{-12},\ 6.32121 \times 10^{-12})$	.035
$a = 5 \times 10^{-7}$	$D_4(T)$	6.32121×10^{-1}	$(6.32109 \times 10^{-1},\ 6.32121 \times 10^{-1})$	.002
$\lambda_1 = 3 \times 10^{-6}$				
$\lambda_2 = 2 \times 10^{-6}$	$D_2(T)$	5.99991×10^{-19}	$(5.99953 \times 10^{-19},\ 6.00000 \times 10^{-19})$	.006
$\lambda_3 = 1 \times 10^{-2}$	$D_3(T)$	2.99996×10^{-15}	$(2.99977 \times 10^{-15},\ 3.00000 \times 10^{-15})$	.006
$a = 1 \times 10^{-8}$	$D_4(T)$	2.99996×10^{-5}	$(2.99995 \times 10^{-5},\ 3.00000 \times 10^{-5})$	.001
$\lambda_1 = 2 \times 10^{-2}$				
$\lambda_2 = 1 \times 10^{-8}$	$D_2(T)$	1.81269×10^{-12}	$(1.76614 \times 10^{-12},\ 2.00000 \times 10^{-12})$	10.333
$\lambda_3 = 4 \times 10^{-5}$	$D_3(T)$	7.25077×10^{-9}	$(7.06456 \times 10^{-9},\ 8.00000 \times 10^{-9})$	10.333
$a = 1 \times 10^{-3}$	$D_4(T)$	1.81269×10^{-1}	$(1.79322 \times 10^{-1},\ 2.00000 \times 10^{-1})$	10.333
$\lambda_1 = 4 \times 10^{-5}$				
$\lambda_2 = 3 \times 10^{-5}$	$D_2(T)$	1.19976×10^{-14}	$(1.19900 \times 10^{-14},\ 1.20000 \times 10^{-14})$	.020
$\lambda_3 = 4 \times 10^{-5}$	$D_3(T)$	1.59951×10^{-9}	$(1.59867 \times 10^{-14},\ 1.60000 \times 10^{-14})$	.053
$a = 1 \times 10^{-6}$	$D_4(T)$	3.99920×10^{-4}	$(3.99904 \times 10^{-4},\ 4.00000 \times 10^{-4})$	.020

Example 11

The eleventh example is a four-state semi-Markov model with three different exponential transitions.



Example 11: Four-state model with exponential transitions.

The following equation defines the death-state probability for state 3 in example 11:

$$
\begin{aligned}
D_3(T) &= \int_0^T \int_0^{T-x} \int_0^{T-x-y} \lambda_1 e^{-\lambda_1 x} \lambda_2 e^{-\lambda_2 y} \lambda_3 e^{-\lambda_3 z}\, dz\, dy\, dx \\
&= 1 - e^{-\lambda_1 T} - \frac{\lambda_1 (e^{-\lambda_1 T} + e^{-\lambda_2 T})}{-\lambda_1 + \lambda_2} - \frac{\lambda_1 (1 - e^{-\lambda_2 T})}{-\lambda_2 + \lambda_3} + \frac{\lambda_1 \lambda_2 (1 - e^{-\lambda_3 T})}{\lambda_3 (-\lambda_2 + \lambda_3)}
\end{aligned}
$$

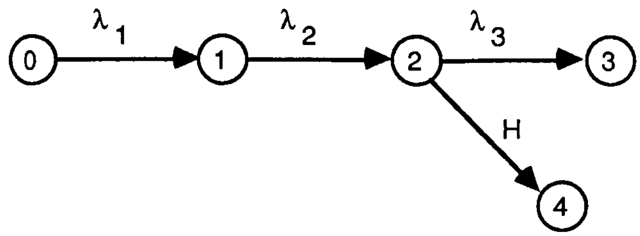
Comparison of SURE With Analytic Solutions for Example 11				
Parameters	Death states	Analytic solutions	SURE bounds	RD
$\lambda_1 = 1 \times 10^{-4}$ $\lambda_2 = 1 \times 10^{-5}$ $\lambda_3 = 1 \times 10^{-6}$	$D_3(T)$	1.66629×10^{-13}	$(1.66620 \times 10^{-13},\ 1.66667 \times 10^{-13})$	0.023
$\lambda_1 = 1 \times 10^{-1}$ $\lambda_2 = 1 \times 10^{-2}$ $\lambda_3 = 1 \times 10^{-3}$	$D_3(T)$	1.28398×10^{-4}	$(1.28398 \times 10^{-4},\ 1.28398 \times 10^{-4})$	.000
$\lambda_1 = 3 \times 10^{-4}$ $\lambda_2 = 2 \times 10^{-4}$ $\lambda_3 = 1 \times 10^{-4}$	$D_3(T)$	9.98501×10^{-10}	$(9.98500 \times 10^{-10},\ 1.00000 \times 10^{-9})$	.150
$\lambda_1 = 6 \times 10^{-6}$ $\lambda_2 = 1 \times 10^{-2}$ $\lambda_3 = 5 \times 10^{-4}$	$D_3(T)$	4.87126×10^{-9}	$(4.86867 \times 10^{-9},\ 5.00000 \times 10^{-9})$	2.643
$\lambda_1 = 3 \times 10^{-1}$ $\lambda_2 = 2 \times 10^{-1}$ $\lambda_3 = 1 \times 10^{-1}$	$D_3(T)$	2.52580×10^{-1}	$(2.52580 \times 10^{-1},\ 2.52580 \times 10^{-1})$	.000
$\lambda_1 = 5 \times 10^{-2}$ $\lambda_2 = 3 \times 10^{-2}$ $\lambda_3 = 2 \times 10^{-2}$	$D_3(T)$	3.90668×10^{-3}	$(3.90668 \times 10^{-3},\ 3.90668 \times 10^{-3})$	.000
$\lambda_1 = 1 \times 10^{-7}$ $\lambda_2 = 1 \times 10^{-4}$ $\lambda_3 = 1 \times 10^{-1}$	$D_3(T)$	1.32086×10^{-10}	$(1.32086 \times 10^{-10},\ 1.32086 \times 10^{-10})$	.000
$\lambda_1 = 4 \times 10^{-2}$ $\lambda_2 = 6 \times 10^{-4}$ $\lambda_3 = 8 \times 10^{-6}$	$D_3(T)$	2.89949×10^{-8}	$(2.89949 \times 10^{-8},\ 2.89949 \times 10^{-8})$	.000
$\lambda_1 = 1 \times 10^{-3}$ $\lambda_2 = 1 \times 10^{-6}$ $\lambda_3 = 1 \times 10^{-7}$	$D_3(T)$	1.66264×10^{-14}	$(1.66250 \times 10^{-14},\ 1.66667 \times 10^{-14})$	.242

Example 12

Example 12 is a five-state semi-Markov model with three exponential transitions and one impulse transition. The impulse distribution $H(t)$ is defined as follows:

$$H(t) = \begin{cases} 0 & (t \leq a) \\ 1 & (t > a) \end{cases}$$

where $a < T$.



Example 12: Five-state model with exponential and impulse transitions.

The following equations define the statistics needed to describe the general transition H in example 12:

$$\rho(H^*) = \int_0^\infty dH(t) = 1$$

$$\mu(H^*) = \frac{1}{\rho(H^*)} \int_0^T t\, dH(t) = a$$

$$\sigma^2(H^*) = \frac{1}{\rho(H^*)} \int_0^T t^2\, d(H)(t) - \mu^2(H^*) = a^2 - a^2 = 0$$

The following equations define the death-state probabilities for states 3 and 4 in example 12:

$$D_3(T) = \int_0^T \int_0^{T-x} \int_0^{T-x-y} \lambda_1 e^{-\lambda_1 x} \lambda_2 e^{-\lambda_2 y} \lambda_3 e^{-\lambda_3 z} [1 - H(z)]\, dz\, dy\, dx$$

$$= \lambda_1 \lambda_2 \lambda_3 \Bigg[\int_0^T e^{-\lambda_1 x} \int_0^{T-x-a} e^{-\lambda_2 y} \int_0^a e^{-\lambda_3 z}\, dz\, dy\, dx$$

$$+ \int_0^T e^{-\lambda_2 y} \int_{T-x-a}^{T-x} e^{-\lambda_2 y} \int_0^{T-x-y} e^{-\lambda_3 z}\, dz\, dy\, dx \Bigg]$$

$$= (1 - e^{-\lambda_3 a})(1 - e^{-\lambda_1 T}) - \frac{\lambda_1 (1 - e^{-\lambda_3 a})[e^{-\lambda_2 a - \lambda_1 T} - e^{-\lambda_2 (T-a)}]}{\lambda_2 - \lambda_1}$$

$$+ \lambda_1 \Bigg[\frac{e^{\lambda_2 a - \lambda_1 T} - e^{-\lambda_2 T + \lambda_2 a} - e^{-\lambda_1 T} + e^{-\lambda_2 T}}{\lambda_2 - \lambda_1}$$

$$+ \frac{e^{-\lambda_1 T + \lambda_3 a} - e^{-\lambda_3 T + \lambda_3 a - \lambda_2 T} - e^{\lambda_3 a + \lambda_2 a - \lambda_1 T} + e^{-(\lambda_2 + \lambda_3)(T-a)}}{-\lambda_1 + \lambda_2 + \lambda_3} \Bigg]$$

$$D_4(T) = \int_0^T \int_0^{T-x} \int_0^{T-x-y} \lambda_1 e^{-\lambda_1 x} \lambda_2 e^{-\lambda_2 y} e^{-\lambda_3 z}\, dH(z)\, dy\, dx$$

$$= \lambda_1 \lambda_2 \int_0^T e^{-\lambda_1} \int_0^{T-x-a} e^{-\lambda_2 y} e^{-\lambda_3 a}\, dy\, dx$$

$$= e^{-\lambda_3 a}(1 - e^{-\lambda_1 T}) - \frac{\lambda_1 e^{-\lambda_3 a}[e^{\lambda_2 a - \lambda_1 T} - e^{-\lambda_2 (T-a)}]}{\lambda_2 - \lambda_1}$$

Comparison of SURE With Analytic Solutions for Example 12				
Parameters	Death states	Analytic solutions	SURE bounds	RD
$\lambda_1 = 5 \times 10^{-5}$ $\lambda_2 = 4 \times 10^{-5}$ $\lambda_3 = 3 \times 10^{-5}$ $a = 1 \times 10^{-7}$	$D_3(T)$ $D_4(T)$	2.99910×10^{-19} 9.99700×10^{-8}	$(2.99820 \times 10^{-19},\ 3.00000 \times 10^{-19})$ $(9.99685 \times 10^{-8},\ 1.00000 \times 10^{-7})$	0.030 .030
$\lambda_1 = 3 \times 10^{-2}$ $\lambda_2 = 2 \times 10^{-2}$ $\lambda_3 = 1 \times 10^{-2}$ $a = 1 \times 10^{-6}$	$D_3(T)$ $D_4(T)$	2.54442×10^{-10} 2.54442×10^{-2}	$(2.54214 \times 10^{-10},\ 2.54442 \times 10^{-10})$ $(2.54426 \times 10^{-2},\ 2.54442 \times 10^{-2})$	.090 .006
$\lambda_1 = 1 \times 10^{-5}$ $\lambda_2 = 1 \times 10^{-3}$ $\lambda_3 = 1 \times 10^{-2}$ $a = 1 \times 10^{-1}$	$D_3(T)$ $D_4(T)$	4.98072×10^{-10} 4.97823×10^{-7}	$(3.63194 \times 10^{-10},\ 5.00000 \times 10^{-10})$ $(4.28464 \times 10^{-7},\ 5.00000 \times 10^{-7})$	27.080 13.932
$\lambda_1 = 4 \times 10^{-4}$ $\lambda_2 = 3 \times 10^{-4}$ $\lambda_3 = 1 \times 10^{-4}$ $a = 1 \times 10^{-2}$	$D_3(T)$ $D_4(T)$	5.98602×10^{-12} 5.98601×10^{-6}	$(5.43629 \times 10^{-12},\ 6.00000 \times 10^{-12})$ $(5.79951 \times 10^{-6},\ 6.00000 \times 10^{-6})$	9.184 3.116
$\lambda_1 = 1 \times 10^{-1}$ $\lambda_2 = 1 \times 10^{-4}$ $\lambda_3 = 1 \times 10^{-7}$ $a = 1 \times 10^{-5}$	$D_3(T)$ $D_4(T)$	3.67749×10^{-16} 3.67747×10^{-4}	$(3.66749 \times 10^{-16},\ 3.67747 \times 10^{-16})$ $(3.67645 \times 10^{-4},\ 3.67747 \times 10^{-4})$	.272 .028
$\lambda_1 = 6 \times 10^{-7}$ $\lambda_2 = 5 \times 10^{-7}$ $\lambda_3 = 1 \times 10^{-1}$ $a = 1 \times 10^{-4}$	$D_3(T)$ $D_4(T)$	1.49999×10^{-16} 1.50000×10^{-11}	$(1.48580 \times 10^{-16},\ 1.50000 \times 10^{-16})$ $(1.49779 \times 10^{-11},\ 1.50000 \times 10^{-11})$	.946 .147
$\lambda_1 = 3 \times 10^{-5}$ $\lambda_2 = 2 \times 10^{-6}$ $\lambda_3 = 4 \times 10^{-7}$ $a = 1$	$D_3(T)$ $D_4(T)$	1.19987×10^{-15} 2.99968×10^{-9}	$(3.83605 \times 10^{-16},\ 1.20000 \times 10^{-15})$ $(1.37616 \times 10^{-9},\ 3.00000 \times 10^{-9})$	68.029 54.123
$\lambda_1 = 5 \times 10^{-6}$ $\lambda_2 = 4 \times 10^{-6}$ $\lambda_3 = 5 \times 10^{-6}$ $a = 4 \times 10^{-6}$	$D_3(T)$ $D_4(T)$	1.99994×10^{-20} 9.99970×10^{-10}	$(1.99615 \times 10^{-20},\ 2.00000 \times 10^{-20})$ $(9.99799 \times 10^{-10},\ 1.00000 \times 10^{-9})$	.190 .017
$\lambda_1 = 3 \times 10^{-3}$ $\lambda_2 = 2 \times 10^{-3}$ $\lambda_3 = 1 \times 10^{-3}$ $a = 1 \times 10^{1}$	$D_3(T)$ $D_4(T)$	2.93577×10^{-6} 2.92111×10^{-4}	$(0.00000,\ 3.00000 \times 10^{-6})$*† $(7.32879 \times 10^{-6},\ 3.00000 \times 10^{-4})$	100.000 97.491

Comparison of SURE With Analytic Solutions for Example 12—Concluded				
Parameters	Death states	Analytic solutions	SURE bounds	RD
$\lambda_1 = 1 \times 10^{-4}$ $\lambda_2 = 1 \times 10^{-3}$ $\lambda_3 = 1 \times 10^{-2}$ $a = 2$	$D_3(T)$ $D_4(T)$	9.86445×10^{-8} 4.88307×10^{-6}	$(1.65525 \times 10^{-8},\ 1.00000 \times 10^{-7})$ $(1.23558 \times 10^{-6},\ 5.00000 \times 10^{-6})$	83.220 74.697
$\lambda_1 = 1 \times 10^{-4}$ $\lambda_2 = 1 \times 10^{-3}$ $\lambda_3 = 1 \times 10^{-2}$ $a = 3$	$D_3(T)$ $D_4(T)$	1.47232×10^{-7} 4.83448×10^{-6}	$(1.33948 \times 10^{-8},\ 1.50000 \times 10^{-7})$* $(6.50732 \times 10^{-7},\ 5.00000 \times 10^{-6})$	90.902 86.540
*RECOVERY TOO SLOW †DELTA > TIME				

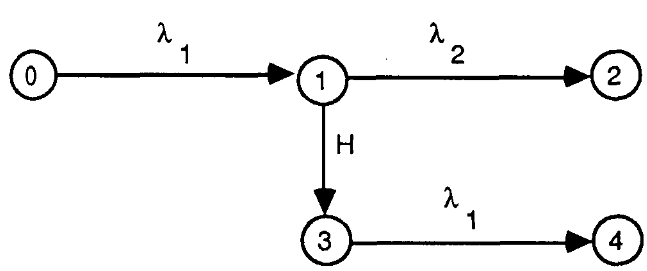
As shown in the table for example 12, the bounds again tended to separate when the recovery transition was slow with respect to the mission time.

Example 13

The thirteenth example is a five-state semi-Markov model with two exponential transitions and one impulse transition. The impulse distribution $H(t)$ is defined as follows:

$$H(t) = \begin{cases} 0 & (t \le a) \\ 1 & (t > a) \end{cases}$$

where $a < T$.



Example 13: Five-state model with exponential and impulse transitions.

The following equations define the statistics needed to describe the general transition H in example 13:

$$\rho(H^*) = \int_0^\infty dH(t) = 1$$

$$\mu(H^*) = \frac{1}{\rho(H^*)} \int_0^\infty t\, dH(t) = a$$

$$\sigma^2(H^*) = \frac{1}{\rho(H^*)} \int_0^\infty t^2\, dH(t) - \mu^2(H^*) = a^2 - a^2 = 0$$

The following equations define the death-state probabilities for states 2 and 4 in example 13:

$$\begin{aligned} D_2(T) &= \int_0^T \int_0^{T-x} \lambda_1 e^{-\lambda_1 x} \lambda_2 e^{-\lambda_2 y} [1 - H(y)] \, dy \, dx \\ &= \lambda_1 \lambda_2 \int_0^{T-a} e^{-\lambda_1 x} \int_0^a e^{-\lambda_2 y} \, dy \, dx + \lambda_1 \lambda_2 \int_{T-a}^T e^{-\lambda_1 x} \int_0^{T-x} e^{-\lambda_2 y} \, dy \, dx \\ &= (1 - e^{-\lambda_2 a})[1 - e^{-\lambda_1 (T-a)}] + e^{-\lambda_1 T} - \frac{\lambda_1 e^{-\lambda_2 T}}{\lambda_2 - \lambda_1} [e^{(\lambda_2 - \lambda_1)T} - e^{(\lambda_2 - \lambda_1)(T-a)}] \end{aligned}$$

$$\begin{aligned} D_4(T) &= \int_0^T \int_0^{T-x} \int_0^{T-x-y} \lambda_1 e^{-\lambda_1 x} e^{-\lambda_2 y} \lambda_1 e^{-\lambda_1 z} \, dz \, dH(y) \, dx \\ &= \lambda_1 \int_0^{T-a} e^{-\lambda_1 x} \{ e^{-\lambda_2 a} [1 - e^{-\lambda_1 (T-x-a)}] \} \, dx \\ &= e^{-\lambda_2 a} - e^{-\lambda_2 a - \lambda_1 (T-a)} - \lambda_1 (T-a) e^{-\lambda_2 a - \lambda_1 (T-a)} \end{aligned}$$

Comparison of SURE With Analytic Solutions for Example 13				
Parameters	Death states	Analytic solutions	SURE bounds	RD
$\lambda_1 = 1 \times 10^{-5}$ $\lambda_2 = 1 \times 10^{-4}$ $a = 1 \times 10^{-6}$	$D_2(T)$ $D_4(T)$	9.99950×10^{-15} 4.99967×10^{-9}	$(9.99318 \times 10^{-15},\ 1.00000 \times 10^{-14})$ $(4.99933 \times 10^{-9},\ 5.00000 \times 10^{-9})$	0.063 .007
$\lambda_1 = 3 \times 10^{-4}$ $\lambda_2 = 2 \times 10^{-4}$ $a = 1 \times 10^{-7}$	$D_2(T)$ $D_4(T)$	5.99101×10^{-14} 4.49101×10^{-6}	$(5.98980 \times 10^{-14},\ 6.00000 \times 10^{-14})$ $(4.49093 \times 10^{-6},\ 4.50000 \times 10^{-6})$	.020 .200
$\lambda_1 = 2 \times 10^{-3}$ $\lambda_2 = 1 \times 10^{-2}$ $a = 4 \times 10^{-5}$	$D_2(T)$ $D_4(T)$	7.92051×10^{-9} 1.97358×10^{-4}	$(7.88851 \times 10^{-9},\ 8.00000 \times 10^{-9})$ $(1.97178 \times 10^{-4},\ 2.00000 \times 10^{-4})$	1.003 1.342
$\lambda_1 = 1 \times 10^{-2}$ $\lambda_2 = 1 \times 10^{-5}$ $a = 1 \times 10^{-3}$	$D_2(T)$ $D_4(T)$	9.51581×10^{-10} 4.67794×10^{-3}	$(9.31585 \times 10^{-10},\ 1.00000 \times 10^{-9})$ $(4.63597 \times 10^{-3},\ 5.00000 \times 10^{-3})$	5.083 6.792
$\lambda_1 = 5 \times 10^{-7}$ $\lambda_2 = 3 \times 10^{-7}$ $a = 1 \times 10^{-1}$	$D_2(T)$ $D_4(T)$	1.49250×10^{-13} 1.22512×10^{-11}	$(1.21500 \times 10^{-13},\ 1.50000 \times 10^{-13})$ $(1.07567 \times 10^{-11},\ 1.25000 \times 10^{-11})$	19.000 12.190
$\lambda_1 = 1 \times 10^{-1}$ $\lambda_2 = 1 \times 10^{-1}$ $a = 1 \times 10^{-6}$	$D_2(T)$ $D_4(T)$	6.32121×10^{-8} 2.64241×10^{-1}	$(6.31804 \times 10^{-8},\ 6.32121 \times 10^{-8})$ $(2.64228 \times 10^{-1},\ 2.64241 \times 10^{-1})$	.050 .005
$\lambda_1 = 3 \times 10^{-2}$ $\lambda_2 = 2 \times 10^{-2}$ $a = 2 \times 10^{-2}$	$D_2(T)$ $D_4(T)$	1.03563×10^{-4} 3.67883×10^{-2}	$(9.51930 \times 10^{-5},\ 1.03673 \times 10^{-4})$ $(3.52419 \times 10^{-2},\ 3.69363 \times 10^{-2})$	8.161 4.203
$\lambda_1 = 4 \times 10^{-4}$ $\lambda_2 = 3 \times 10^{-4}$ $a = 5$	$D_2(T)$ $D_4(T)$	4.49001×10^{-6} 1.99434×10^{-6}	$(5.13100 \times 10^{-7},\ 6.00000 \times 10^{-6})$* $(2.04734 \times 10^{-7},\ 8.00000 \times 10^{-6})$	91.425 100.000
$\lambda_1 = 4 \times 10^{-4}$ $\lambda_2 = 3 \times 10^{-4}$ $a = 3$	$D_2(T)$ $D_4(T)$	3.05346×10^{-6} 3.90917×10^{-6}	$(7.34999 \times 10^{-7},\ 3.60000 \times 10^{-6})$* $(1.08570 \times 10^{-6},\ 8.00000 \times 10^{-6})$	79.533 150.617
$\lambda_1 = 4 \times 10^{-4}$ $\lambda_2 = 3 \times 10^{-4}$ $a = 2$	$D_2(T)$ $D_4(T)$	2.15548×10^{-6} 5.10603×10^{-6}	$(7.32166 \times 10^{-7},\ 2.40000 \times 10^{-6})$ $(2.02824 \times 10^{-6},\ 8.00000 \times 10^{-6})$	69.423 67.108
$\lambda_1 = 3 \times 10^{-3}$ $\lambda_2 = 1 \times 10^{-3}$ $a = 1$	$D_2(T)$ $D_4(T)$	2.80835×10^{-5} 3.57647×10^{-4}	$(1.38723 \times 10^{-5},\ 3.00000 \times 10^{-5})$ $(2.03197 \times 10^{-4},\ 4.50000 \times 10^{-4})$	53.038 42.482
$\lambda_1 = 3 \times 10^{-5}$ $\lambda_2 = 2 \times 10^{-5}$ $a = 5 \times 10^{-2}$	$D_2(T)$ $D_4(T)$	2.99205×10^{-10} 4.45422×10^{-8}	$(2.59037 \times 10^{-10},\ 3.00000 \times 10^{-10})$ $(4.09686 \times 10^{-8},\ 4.50000 \times 10^{-8})$	13.641 8.021
*RECOVERY TOO SLOW				

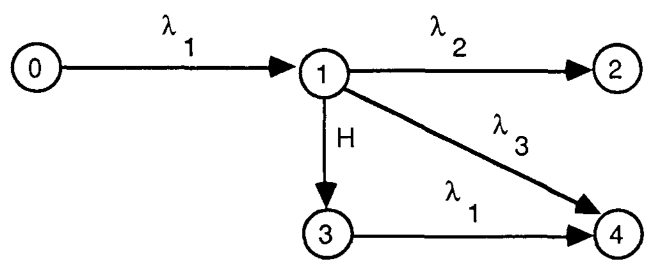
Note in the table for example 13 that large relative errors occurred when the recovery rate was slow.

Example 14

The fourteenth example is a five-state semi-Markov model like that for example 13 except that there is an additional exponential transition from state 1 to state 4. The impulse distribution H is defined as follows:

$$H(t) = \begin{cases} 0 & (t \leq a) \\ 1 & (t > a) \end{cases}$$

where $a < T$.



Example 14: Five-state model with exponential and impulse transitions.

The following equations define the statistics needed to describe the general transition H in example 14:

$$\rho(H^*) = \int_0^\infty dH(t) = 1$$

$$\mu(H^*) = \frac{1}{\rho(H^*)} \int_0^\infty t\, dH(t) = a$$

$$\sigma^2(H^*) = \frac{1}{\rho(H^*)} \int_0^\infty t^2\, dH(t) - \mu^2(H^*) = a^2 - a^2 = 0$$

The following equations define the death-state probabilities for states 2 and 4 in example 14:

$$D_2(T) = \int_0^T \int_0^{T-x} \lambda_1 e^{-\lambda_1 x} \lambda_2 e^{-(\lambda_2+\lambda_3)y} [1 - H(y)]\, dy\, dx$$

$$= \lambda_1 \lambda_2 \int_0^T e^{-\lambda_1 x} \int_0^a e^{-(\lambda_2+\lambda_3)y}\, dy\, dx = \frac{\lambda_2 [1 - e^{-(\lambda_2+\lambda_3)a}](1 - e^{-\lambda_1 T})}{\lambda_2 + \lambda_3} .$$

$$D_4(T) = \int_0^T \int_0^{T-x} \lambda_1 e^{-\lambda_1 x} \lambda_3 e^{-(\lambda_2+\lambda_3)y} [1 - H(y)]\, dy\, dx$$

$$+ \int_0^T \int_0^{T-x} \int_0^{T-x-y} \lambda_1 e^{-\lambda_1 x} e^{-(\lambda_2+\lambda_3)y} \lambda_1 e^{-\lambda_1 z}\, dz\, dH(y)\, dx$$

$$= \frac{\lambda_3 [1 - e^{-(\lambda_2+\lambda_3)a}](1 - e^{-\lambda_1 T})}{\lambda_2 + \lambda_3} + \lambda_1 e^{-(\lambda_2+\lambda_3)a} \frac{1 - e^{-\lambda_1 T}}{\lambda_1 - T e^{-\lambda_1 (T-a)}}$$

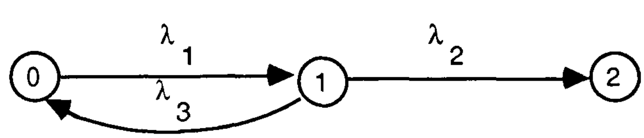
Comparison of SURE With Analytic Solutions for Example 14				
Parameters	Death states	Analytic solutions	SURE bounds	RD
$\lambda_1 = 4 \times 10^{-3}$, $\lambda_2 = 3 \times 10^{-3}$	$D_2(T)$	1.17632×10^{-9}	$(1.17367 \times 10^{-9},\ 1.20000 \times 10^{-9})$	2.013
$\lambda_3 = 2 \times 10^{-3}$, $a = 1 \times 10^{-5}$	$D_4(T)$	7.78982×10^{-4}	$(7.78425 \times 10^{-4},\ 8.00001 \times 10^{-4})$	2.698
$\lambda_1 = 3 \times 10^{-2}$, $\lambda_2 = 2 \times 10^{-2}$	$D_2(T)$	5.18363×10^{-8}	$(5.17401 \times 10^{-8},\ 5.18364 \times 10^{-8})$	.186
$\lambda_3 = 1 \times 10^{-1}$, $a = 1 \times 10^{-5}$	$D_4(T)$	3.69365×10^{-2}	$(3.69258 \times 10^{-2},\ 3.69366 \times 10^{-2})$	.029
$\lambda_1 = 5 \times 10^{-5}$, $\lambda_2 = 4 \times 10^{-5}$	$D_2(T)$	1.99949×10^{-11}	$(1.95970 \times 10^{-11},\ 2.00000 \times 10^{-11})$	1.990
$\lambda_3 = 1 \times 10^{-2}$, $a = 1 \times 10^{-3}$	$D_4(T)$	1.29931×10^{-7}	$(1.29010 \times 10^{-7},\ 1.30000 \times 10^{-7})$	.709
$\lambda_1 = 1 \times 10^{-3}$, $\lambda_2 = 1 \times 10^{-3}$	$D_2(T)$	4.97508×10^{-11}	$(4.96798 \times 10^{-11},\ 5.00000 \times 10^{-11})$	.501
$\lambda_3 = 1 \times 10^{-6}$, $a = 5 \times 10^{-6}$	$D_4(T)$	4.96679×10^{-5}	$(4.96568 \times 10^{-5},\ 5.00000 \times 10^{-5})$	.669
$\lambda_1 = 2 \times 10^{-3}$, $\lambda_2 = 3 \times 10^{-6}$	$D_2(T)$	1.18808×10^{-12}	$(1.18466 \times 10^{-12},\ 1.20000 \times 10^{-12})$	1.003
$\lambda_3 = 1 \times 10^{-4}$, $a = 2 \times 10^{-5}$	$D_4(T)$	1.97353×10^{-4}	$(1.97235 \times 10^{-4},\ 2.00000 \times 10^{-4})$	1.342
$\lambda_1 = 1 \times 10^{-6}$, $\lambda_2 = 1 \times 10^{-1}$	$D_2(T)$	3.99916×10^{-9}	$(3.84078 \times 10^{-9},\ 4.00000 \times 10^{-9})$	3.960
$\lambda_3 = 2 \times 10^{-3}$, $a = 4 \times 10^{-3}$	$D_4(T)$	1.29923×10^{-10}	$(1.25945 \times 10^{-10},\ 1.30000 \times 10^{-10})$	3.062
$\lambda_1 = 8 \times 10^{-6}$, $\lambda_2 = 7 \times 10^{-6}$	$D_2(T)$	2.79989×10^{-15}	$(2.79593 \times 10^{-15},\ 2.80000 \times 10^{-15})$	.141
$\lambda_3 = 1 \times 10^{-2}$, $a = 5 \times 10^{-6}$	$D_4(T)$	3.20383×10^{-9}	$(3.20319 \times 10^{-9},\ 3.20400 \times 10^{-9})$	.020
$\lambda_1 = 3 \times 10^{-2}$, $\lambda_2 = 2 \times 10^{-2}$	$D_2(T)$	5.13212×10^{-3}	$(2.49914 \times 10^{-3},\ 5.18364 \times 10^{-3})$	51.304
$\lambda_3 = 1 \times 10^{-5}$, $a = 1$	$D_4(T)$	2.95728×10^{-2}	$(1.74563 \times 10^{-2},\ 3.69389 \times 10^{-2})$	40.972
$\lambda_1 = 1 \times 10^{-3}$, $\lambda_2 = 1 \times 10^{-2}$	$D_2(T)$	9.89564×10^{-6}	$(8.01427 \times 10^{-6},\ 1.00000 \times 10^{-5})$	19.012
$\lambda_3 = 1 \times 10^{-1}$, $a = 1 \times 10^{-1}$	$D_4(T)$	1.47102×10^{-4}	$(1.22415 \times 10^{-4},\ 1.50000 \times 10^{-4})$	16.782

Comparison of SURE With Analytic Solutions for Example 14—Concluded				
Parameters	Death states	Analytic solutions	SURE bounds	RD
$\lambda_1 = 6 \times 10^{-4}$ $\lambda_2 = 5 \times 10^{-4}$ $\lambda_3 = 3 \times 10^{-2}$ $a = 1 \times 10^{-8}$	$D_2(T)$	2.99102×10^{-14}	$(2.99081 \times 10^{-14},\ 3.00000 \times 10^{-14})$	0.300
	$D_4(T)$	1.79282×10^{-5}	$(1.79279 \times 10^{-5},\ 1.80000 \times 10^{-5})$	.400
$\lambda_1 = 1 \times 10^{-6}$ $\lambda_2 = 1 \times 10^{-5}$ $\lambda_3 = 1 \times 10^{-8}$ $a = 1 \times 10^{-1}$	$D_2(T)$	9.99994×10^{-12}	$(8.09996 \times 10^{-12},\ 1.00000 \times 10^{-11})$	19.000
	$D_4(T)$	4.90096×10^{-11}	$(4.30347 \times 10^{-11},\ 5.00100 \times 10^{-11})$	12.191
$\lambda_1 = 5 \times 10^{-5}$ $\lambda_2 = 4 \times 10^{-5}$ $\lambda_3 = 1 \times 10^{-2}$ $a = 2$	$D_2(T)$	3.95912×10^{-8}	$(1.19993 \times 10^{-8},\ 4.00000 \times 10^{-8})$	69.692
	$D_4(T)$	9.97128×10^{-6}	$(3.03076 \times 10^{-6},\ 1.01250 \times 10^{-5})$	69.605
$\lambda_1 = 5 \times 10^{-5}$ $\lambda_2 = 4 \times 10^{-5}$ $\lambda_3 = 1 \times 10^{-2}$ $a = 3$	$D_2(T)$	5.90906×10^{-8}	$(1.18633 \times 10^{-8},\ 6.00000 \times 10^{-8})$*	79.924
	$D_4(T)$	1.48212×10^{-5}	$(2.98211 \times 10^{-6},\ 1.51250 \times 10^{-5})$	79.879
*RECOVERY TOO SLOW				

Again, for example 14 large relative errors occurred when recovery times were relatively slow with respect to the mission time.

Example 15

The fifteenth example is a three-state semi-Markov model with a transient fault where all transitions are exponential.



Example 15: Three-state model with a transient fault.

Since there are an infinite number of paths leading to the death state in this model, an exact expression for the unreliability cannot be obtained by using convolution integrals. However, this model is pure Markov; so, an exact expression for the death-state probability $D_2(T)$ can be obtained by solving the following set of differential equations:

$$D_0'(T) = -\lambda_1 D_0(T) + \lambda_3 D_1(T)$$

$$D_1'(T) = \lambda_1 D_0(T) - (\lambda_2 + \lambda_3) D_1(T)$$

$$D_2'(T) = \lambda_2 D_1(T)$$

$$D_2(T) = \left\{ \frac{e^{[(\lambda_1+\lambda_2+\lambda_3)^2-4\lambda_1\lambda_2]^{1/2}T/2}}{-(\lambda_1+\lambda_2+\lambda_3)[(\lambda_1+\lambda_2+\lambda_3)^2-4\lambda_1\lambda_2]^{1/2}+(\lambda_1+\lambda_2+\lambda_3)^2-4\lambda_1\lambda_2} - \frac{e^{-[(\lambda_1+\lambda_2+\lambda_3)^2-4\lambda_1\lambda_2]^{1/2}T/2}}{-(\lambda_1+\lambda_2+\lambda_3)[(\lambda_1+\lambda_2+\lambda_3)^2-4\lambda_1\lambda_2]^{1/2}-(\lambda_1+\lambda_2+\lambda_3)^2+4\lambda_1\lambda_2} \right\} 2\lambda_1\lambda_2 e^{-(\lambda_1+\lambda_2+\lambda_3)T/2}$$

Comparison of SURE With Analytic Solutions for Example 15

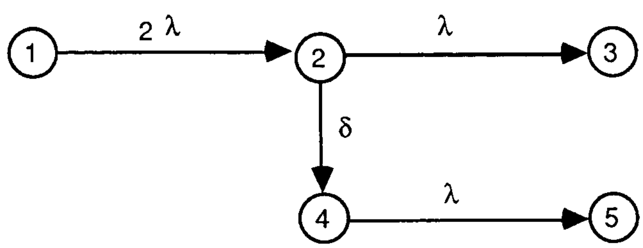
Parameters	Death states	Analytic solutions	SURE bounds	RD
$\lambda_1 = 5 \times 10^{-4}$ $\lambda_2 = 4 \times 10^{-4}$ $\lambda_3 = 3 \times 10^{-4}$	$D_2(T)$	9.96009×10^{-6}	$(9.96001 \times 10^{-6},\ 1.00000 \times 10^{-5})$	0.401
$\lambda_1 = 1 \times 10^{-3}$ $\lambda_2 = 1 \times 10^{-5}$ $\lambda_3 = 1 \times 10^{-7}$	$D_2(T)$	4.98321×10^{-7}	$(4.98317 \times 10^{-7},\ 5.00000 \times 10^{-7})$	.337
$\lambda_1 = 1 \times 10^{-7}$ $\lambda_2 = 1 \times 10^{-6}$ $\lambda_3 = 1 \times 10^{-5}$	$D_2(T)$	4.99982×10^{-12}	$(4.99981 \times 10^{-12},\ 5.00000 \times 10^{-12})$	.004
$\lambda_1 = 3 \times 10^{-4}$ $\lambda_2 = 2 \times 10^{-4}$ $\lambda_3 = 1 \times 10^{-1}$	$D_2(T)$	2.20417×10^{-6}	$(2.20417 \times 10^{-6},\ 2.20417 \times 10^{-6})$	.000
$\lambda_1 = 1 \times 10^{-5}$ $\lambda_2 = 1 \times 10^{-7}$ $\lambda_3 = 1 \times 10^{-3}$	$D_2(T)$	4.98321×10^{-11}	$(4.98317 \times 10^{-11},\ 5.00000 \times 10^{-11})$	.337
$\lambda_1 = 8 \times 10^{-4}$ $\lambda_2 = 4 \times 10^{-4}$ $\lambda_3 = 3 \times 10^{-4}$	$D_2(T)$	1.59202×10^{-5}	$(1.59200 \times 10^{-5},\ 1.60000 \times 10^{-5})$	.501
$\lambda_1 = 1 \times 10^{-1}$ $\lambda_2 = 1 \times 10^{-2}$ $\lambda_3 = 1 \times 10^{-3}$	$D_2(T)$	3.54027×10^{-2}	$(3.54027 \times 10^{-2},\ 3.54027 \times 10^{-2})$	.000
$\lambda_1 = 2 \times 10^{-3}$ $\lambda_2 = 3 \times 10^{-1}$ $\lambda_3 = 2 \times 10^{-4}$	$D_2(T)$	1.35515×10^{-2}	$(1.35515 \times 10^{-2},\ 1.35515 \times 10^{-2})$	.000
$\lambda_1 = 5 \times 10^{-2}$ $\lambda_2 = 4 \times 10^{-2}$ $\lambda_3 = 1 \times 10^{-7}$	$D_2(T)$	7.45224×10^{-2}	$(7.45224 \times 10^{-2},\ 7.45224 \times 10^{-2})$	.000
$\lambda_1 = 3 \times 10^{-3}$ $\lambda_2 = 2 \times 10^{-3}$ $\lambda_3 = 1 \times 10^{-6}$	$D_2(T)$	2.95046×10^{-4}	$(2.94999 \times 10^{-4},\ 3.00000 \times 10^{-4})$	1.679

Appendix B

Comparison of SURE With Other Reliability Analysis Tools

Example 16

The model for example 16 represents a duplex system which could consist of two similar components such as processors or buses. Only one working component is necessary for the system to function properly. The component failure rate is given by λ and the recovery rate is denoted by δ.

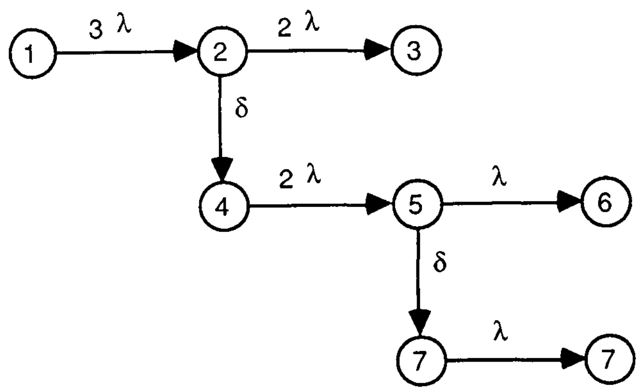


Example 16: Duplex system with permanent faults.

Comparison Among SURE, PAWS, and CARE III for Example 16				
Parameters	Death state	SURE bounds	PAWS	CARE III
$\lambda = 1 \times 10^{-2}$	$D_3(T)$	$(1.65059 \times 10^{-5},\ 2.00000 \times 10^{-5})$	1.81087×10^{-5}	
$\delta = 1 \times 10^{2}$	$D_5(T)$	$(8.66173 \times 10^{-3},\ 1.00000 \times 10^{-2})$	9.03781×10^{-3}	
	Total	$(8.67824 \times 10^{-3},\ 1.00200 \times 10^{-2})$	9.05592×10^{-3}	9.05591×10^{-3}
$\lambda = 1 \times 10^{-2}$	$D_3(T)$	$(1.78482 \times 10^{-7},\ 2.00000 \times 10^{-7})$	1.81267×10^{-7}	
$\delta = 1 \times 10^{4}$	$D_5(T)$	$(8.98416 \times 10^{-3},\ 1.00000 \times 10^{-2})$	9.05574×10^{-3}	
	Total	$(8.98434 \times 10^{-3},\ 1.00002 \times 10^{-2})$	9.05592×10^{-3}	9.05591×10^{-3}
$\lambda = 1 \times 10^{-5}$	$D_3(T)$	$(1.99962 \times 10^{-17},\ 2.00000 \times 10^{-17})$	1.99980×10^{-17}	
$\delta = 1 \times 10^{8}$	$D_5(T)$	$(9.99896 \times 10^{-9},\ 1.00000 \times 10^{-8})$	9.99900×10^{-9}	
	Total	$(9.99896 \times 10^{-9},\ 1.00000 \times 10^{-8})$	9.99900×10^{-9}	9.99900×10^{-9}
$\lambda = 1 \times 10^{-2}$	$D_3(T)$	$(1.75218 \times 10^{-6},\ 2.00000 \times 10^{-6})$	1.81251×10^{-6}	
$\delta = 1 \times 10^{3}$	$D_5(T)$	$(8.92656 \times 10^{-3},\ 1.00000 \times 10^{-2})$	9.05410×10^{-3}	
	Total	$(8.92831 \times 10^{-3},\ 1.00200 \times 10^{-2})$	9.05592×10^{-3}	9.05591×10^{-3}
$\lambda = 1$	$D_3(T)$	$(9.99552 \times 10^{-7},\ 1.00000 \times 10^{-6})$	9.99999×10^{-7}	
$\delta = 1 \times 10^{6}$	$D_5(T)$	$(9.99891 \times 10^{-1},\ 9.99909 \times 10^{-1})$	9.99908×10^{-1}	
	Total	$(9.99892 \times 10^{-1},\ 9.99910 \times 10^{-1})$	9.99909×10^{-1}	9.99909×10^{-1}
$\lambda = 1 \times 10^{-4}$	$D_3(T)$	$(1.47303 \times 10^{-8},\ 2.00000 \times 10^{-8})$	1.97802×10^{-8}	
$\delta = 1 \times 10^{1}$	$D_5(T)$	$(8.25682 \times 10^{-7},\ 1.00000 \times 10^{-6})$	9.79220×10^{-7}	
	Total	$(8.40412 \times 10^{-7},\ 1.02000 \times 10^{-6})$	9.99001×10^{-7}	9.99000×10^{-7}

Example 17

Example 17 is similar to example 16 but has one additional component; hence, example 17 models a triad with permanent faults. Again, only one component is necessary for the system to operate. The permanent fault-arrival rate is given by λ and the fault-recovery rate is given by δ.



Example 17: Triad with permanent faults.

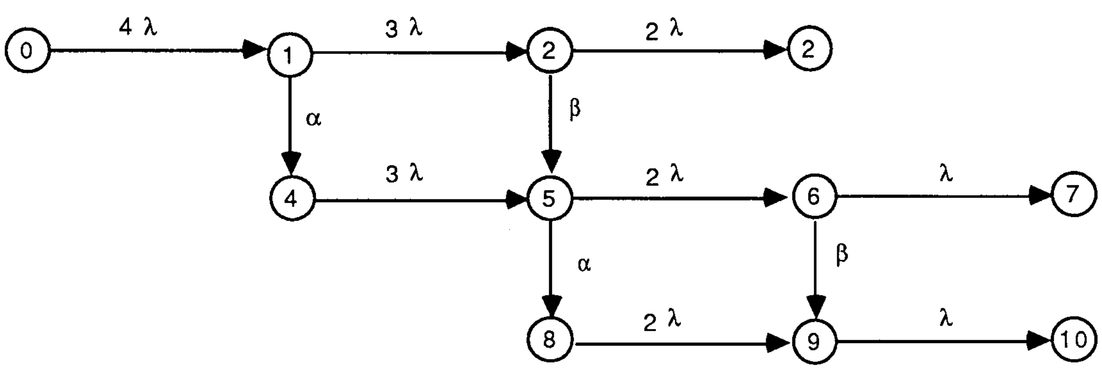
Comparison Among SURE, PAWS, and CARE III for Example 17				
Parameters	Death state	SURE bounds	PAWS	CARE III
$\lambda = 1 \times 10^{-3}$	$D_3(T)$	$(5.39677 \times 10^{-7},\ 6.00000 \times 10^{-7})$	5.90495×10^{-7}	
$\delta = 1 \times 10^{2}$	$D_6(T)$	$(2.46916 \times 10^{-9},\ 3.00000 \times 10^{-9})$	2.93870×10^{-9}	
	$D_8(T)$	$(8.80393 \times 10^{-7},\ 1.00000 \times 10^{-6})$	9.79231×10^{-7}	
	Total	$(1.42254 \times 10^{-6},\ 1.60300 \times 10^{-6})$	1.57267×10^{-6}	1.57588×10^{-6}
$\lambda = 1 \times 10^{-2}$	$D_3(T)$	$(5.04812 \times 10^{-6},\ 5.18364 \times 10^{-6})$	5.18309×10^{-6}	
$\delta = 1 \times 10^{3}$	$D_6(T)$	$(2.42381 \times 10^{-7},\ 2.54442 \times 10^{-7})$	2.54341×10^{-7}	
	$D_8(T)$	$(8.42161 \times 10^{-4},\ 8.61784 \times 10^{-4})$	8.61267×10^{-4}	
	Total	$(8.47451 \times 10^{-4},\ 8.67222 \times 10^{-4})$	8.66705×10^{-4}	8.66969×10^{-4}
$\lambda = 1 \times 10^{-4}$	$D_3(T)$	$(5.99046 \times 10^{-15},\ 6.00000 \times 10^{-15})$	5.99101×10^{-15}	
$\delta = 1 \times 10^{8}$	$D_6(T)$	$(2.99459 \times 10^{-18},\ 3.00000 \times 10^{-18})$	2.99500×10^{-18}	
	$D_8(T)$	$(9.98489 \times 10^{-10},\ 1.00000 \times 10^{-9})$	9.98501×10^{-10}	
	Total	$(9.98495 \times 10^{-10},\ 1.00000 \times 10^{-9})$	9.98507×10^{-10}	9.98501×10^{-10}
$\lambda = 1 \times 10^{-8}$	$D_3(T)$	$(5.47534 \times 10^{-17},\ 6.00000 \times 10^{-17})$	5.99400×10^{-17}	
$\delta = 1 \times 10^{2}$	$D_6(T)$	$(2.50897 \times 10^{-24},\ 3.00000 \times 10^{-24})$	2.98802×10^{-24}	
	$D_8(T)$	$(8.93395 \times 10^{-22},\ 1.00000 \times 10^{-21})$	9.94018×10^{-22}	
	Total	$(5.47543 \times 10^{-17},\ 6.00010 \times 10^{-17})$	5.99410×10^{-17}	5.99676×10^{-17}
$\lambda = 1 \times 10^{-6}$	$D_3(T)$	$(5.99454 \times 10^{-17},\ 6.00000 \times 10^{-17})$	5.99991×10^{-17}	
$\delta = 1 \times 10^{6}$	$D_6(T)$	$(2.99567 \times 10^{-22},\ 3.00000 \times 10^{-22})$	2.99995×10^{-22}	
	$D_8(T)$	$(9.99746 \times 10^{-16},\ 1.00000 \times 10^{-15})$	9.99984×10^{-16}	
	Total	$(1.05969 \times 10^{-15},\ 1.06000 \times 10^{-15})$	1.05998×10^{-15}	1.06019×10^{-15}

Example 18

The model for example 18 represents a system of four components. In this model of the system, three faults that occur before the system can recover cause the system to fail; otherwise, the system can function with only one working component. The fault-arrival rate is given by λ; the fault-recovery rate from a first fault is given by α, and from a second fault, by β.

Models that consider critical-triple faults cannot be directly modeled with CARE III since the program does not allow the user to specify M of N gates in its critical-pair fault tree for $M \geq 3$. The CARE III user's guide does state that postprocessing involving the calculation of a convolution integral may be required when critical-triple failures are modeled. CARE III presumably, however, gives a conservative estimate of unreliability in cases with critical-triple faults;[3] this estimate is given in the following table.

From the table for example 18, one can see that although the CARE III solutions for unreliability are conservative, they are several orders of magnitude larger than the SURE and PAWS solutions.



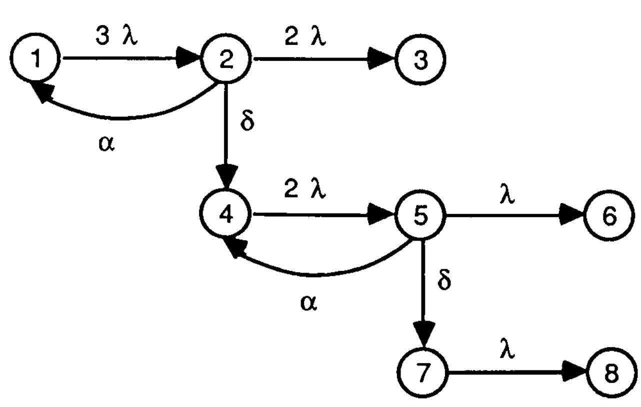
Example 18: Critical-triple four-plex.

[3] There is no formal mathematical proof nor is there documented testing to show that CARE III's estimates are always conservative for models with critical-triple faults. However, since CARE III considers only critically coupled faults when assessing a model that contains critical-triple faults, the total unreliability that CARE III computes for that model should be larger than the true unreliability of the system. In essence, CARE III assumes that the system can tolerate fewer faults than it actually can; hence, the system fails faster and is deemed less reliable.

Comparison Among SURE, PAWS, and CARE III for Example 18				
Parameters	Death state	SURE bounds	PAWS	CARE III
$\lambda = 1 \times 10^{-3}$	$D_3(T)$	$(2.14761 \times 10^{-16},\ 2.40000 \times 10^{-16})$	2.35026×10^{-16}	
$\alpha = 1 \times 10^{2}$	$D_7(T)$	$(9.83067 \times 10^{-19},\ 1.20240 \times 10^{-18})$	1.17003×10^{-18}	
$\beta = 1 \times 10^{7}$	$D_{10}(T)$	$(8.54421 \times 10^{-9},\ 1.00000 \times 10^{-8})$	9.80215×10^{-9}	
	Total	$(8.54421 \times 10^{-9},\ 1.00801 \times 10^{-8})$	9.80215×10^{-9}	1.19718×10^{-6}
$\lambda = 1$	$D_3(T)$	$(5.61088 \times 10^{-9},\ 6.00000 \times 10^{-9})$	5.88235×10^{-9}	
$\alpha = 1 \times 10^{7}$	$D_7(T)$	$(1.89028 \times 10^{-9},\ 2.00000 \times 10^{-9})$	1.98020×10^{-9}	
$\beta = 1 \times 10^{2}$	$D_{10}(T)$	$(9.99811 \times 10^{-1},\ 9.99819 \times 10^{-1})$	9.99818×10^{-1}	
	Total	$(9.99811 \times 10^{-1},\ 9.99819 \times 10^{-1})$	9.99818×10^{-1}	1.00000
$\lambda = 1 \times 10^{-4}$	$D_3(T)$	$(2.30709 \times 10^{-17},\ 2.40000 \times 10^{-17})$	2.39494×10^{-17}	
$\alpha = 1 \times 10^{3}$	$D_7(T)$	$(1.12214 \times 10^{-20},\ 1.20024 \times 10^{-20})$	1.19694×10^{-20}	
$\beta = 1 \times 10^{4}$	$D_{10}(T)$	$(9.68412 \times 10^{-13},\ 1.00080 \times 10^{-12})$	9.98002×10^{-13}	
	Total	$(9.68435 \times 10^{-13},\ 1.00082 \times 10^{-12})$	9.98026×10^{-13}	1.20012×10^{-9}
$\lambda = 1 \times 10^{-5}$	$D_3(T)$	$(2.18908 \times 10^{-22},\ 2.40000 \times 10^{-22})$	2.39712×10^{-22}	
$\alpha = 1 \times 10^{7}$	$D_7(T)$	$(1.04539 \times 10^{-26},\ 1.20000 \times 10^{-26})$	1.19732×10^{-26}	
$\beta = 1 \times 10^{2}$	$D_{10}(T)$	$(9.99736 \times 10^{-17},\ 1.00000 \times 10^{-16})$	9.99800×10^{-17}	
	Total	$(9.99736 \times 10^{-17},\ 1.00000 \times 10^{-16})$	9.99802×10^{-17}	1.19909×10^{-10}
$\lambda = 1 \times 10^{-6}$	$D_3(T)$	$(2.39103 \times 10^{-27},\ 2.40000 \times 10^{-27})$	2.39994×10^{-27}	
$\alpha = 1 \times 10^{5}$	$D_7(T)$	$(1.19281 \times 10^{-32},\ 1.20000 \times 10^{-32})$	1.19936×10^{-32}	
$\beta = 1 \times 10^{6}$	$D_{10}(T)$	$(9.98560 \times 10^{-21},\ 1.00001 \times 10^{-20})$	9.99980×10^{-21}	
	Total	$(9.98560 \times 10^{-21},\ 1.00001 \times 10^{-20})$	9.99980×10^{-21}	1.20310×10^{-15}

Example 19

Example 19 is a triad of components susceptible to both permanent and transient faults. Two faults that occur before recovery can take place cause system failure; otherwise, the system degrades until there are no more functioning components. The fault-arrival rate is given by λ, the permanent-fault-recovery rate is given by δ, and the transient-fault-recovery rate is given by α.



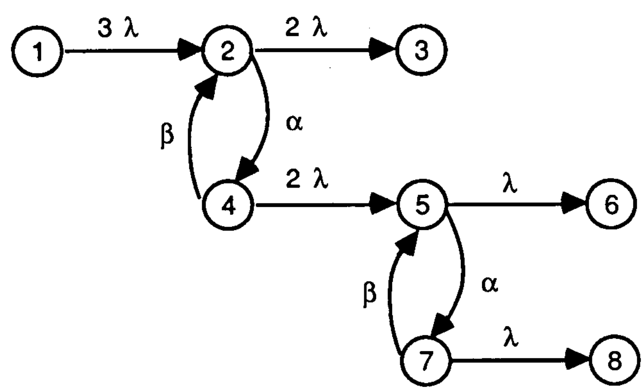
Example 19: Triad with permanent and transient faults.

Comparison Among SURE, PAWS, and CARE III for Example 19				
Parameters	Death state	SURE bounds	PAWS	CARE III
$\lambda = 1 \times 10^{-3}$	$D_3(T)$	$(5.83601 \times 10^{-10}, 5.94148 \times 10^{-10})$	5.83523×10^{-10}	
$\delta = 1 \times 10^{5}$	$D_6(T)$	$(2.87909 \times 10^{-12}, 2.94137 \times 10^{-12})$	2.89280×10^{-12}	
$\alpha = 1 \times 10^{3}$	$D_8(T)$	$(9.64650 \times 10^{-7}, 9.80417 \times 10^{-7})$	9.65827×10^{-7}	
	Total	$(9.65236 \times 10^{-7}, 9.81015 \times 10^{-7})$	9.66415×10^{-7}	9.63961×10^{-7}
$\lambda = 1 \times 10^{-8}$	$D_3(T)$	$(5.99824 \times 10^{-22}, 5.99994 \times 10^{-22})$	5.99994×10^{-22}	
$\delta = 1 \times 10^{7}$	$D_6(T)$	$(2.99861 \times 10^{-29}, 2.99994 \times 10^{-29})$	2.99992×10^{-29}	
$\alpha = 1 \times 10^{2}$	$D_8(T)$	$(9.99928 \times 10^{-22}, 9.99980 \times 10^{-22})$	9.99980×10^{-22}	
	Total	$(1.59975 \times 10^{-21}, 1.59997 \times 10^{-21})$	1.59997×10^{-21}	1.31233×10^{-21}
$\lambda = 1 \times 10^{-4}$	$D_3(T)$	$(5.93752 \times 10^{-11}, 5.99994 \times 10^{-11})$	5.99089×10^{-11}	
$\delta = 1 \times 10^{4}$	$D_6(T)$	$(2.94950 \times 10^{-14}, 2.99994 \times 10^{-14})$	2.99483×10^{-14}	
$\alpha = 1 \times 10^{-1}$	$D_8(T)$	$(9.93342 \times 10^{-10}, 9.99980 \times 10^{-10})$	9.98421×10^{-10}	
	Total	$(1.05275 \times 10^{-9}, 1.06001 \times 10^{-9})$	1.05836×10^{-9}	1.03322×10^{-9}
$\lambda = 1 \times 10^{-6}$	$D_3(T)$	$(5.83135 \times 10^{-14}, 5.99994 \times 10^{-14})$	5.99925×10^{-14}	
$\delta = 1 \times 10^{3}$	$D_6(T)$	$(2.84962 \times 10^{-19}, 2.99994 \times 10^{-19})$	2.99869×10^{-19}	
$\alpha = 1 \times 10^{-2}$	$D_8(T)$	$(9.76239 \times 10^{-16}, 9.99980 \times 10^{-16})$	9.99365×10^{-16}	
	Total	$(5.92900 \times 10^{-14}, 6.09997 \times 10^{-14})$	6.09922×10^{-14}	3.10625×10^{-14}
$\lambda = 1 \times 10^{-4}$	$D_3(T)$	$(5.81714 \times 10^{-10}, 5.99401 \times 10^{-10})$	5.98443×10^{-10}	
$\delta = 1 \times 10^{3}$	$D_6(T)$	$(2.83946 \times 10^{-13}, 2.99401 \times 10^{-13})$	2.98783×10^{-13}	
$\alpha = 1$	$D_8(T)$	$(9.72888 \times 10^{-10}, 9.98004 \times 10^{-10})$	9.95911×10^{-10}	
	Total	$(1.55489 \times 10^{-9}, 1.59771 \times 10^{-9})$	1.59465×10^{-9}	1.29556×10^{-9}
$\lambda = 1 \times 10^{-5}$	$D_3(T)$	$(5.97707 \times 10^{-14}, 5.99490 \times 10^{-14})$	5.99400×10^{-14}	
$\delta = 1 \times 10^{2}$	$D_6(T)$	$(2.98015 \times 10^{-21}, 2.99451 \times 10^{-21})$	2.99400×10^{-21}	
$\alpha = 1 \times 10^{5}$	$D_8(T)$	$(9.96870 \times 10^{-19}, 9.98128 \times 10^{-19})$	9.97972×10^{-19}	
	Total	$(5.97717 \times 10^{-14}, 5.99500 \times 10^{-14})$	5.99410×10^{-14}	1.69753×10^{-14}

Note in the table for example 19 that the unreliability estimates given by CARE III are not conservative.[4] No warning messages or any other indication was given in any of these cases to inform the user that the unreliability estimate may not be conservative. Many attempts were made by varying the run-time parameters of the CARE III program to obtain a conservative estimate from CARE III; but, the estimates were the same regardless of the variation in the run-time parameters.[5] Since CARE III's answers are relatively close to the other answers, CARE III's estimates may be considered good enough for use based on engineering judgment. However, the source of the error is unknown; and, more importantly, it is not known whether the error accumulates as a model of this kind increases in size and complexity. Potential users of CARE III should be advised of this situation and should not expect the program to warn them that the unreliability estimate is not conservative.

Example 20

A triad of components is also the subject for example 20. This model is not a typical construct in modeling fault-tolerant systems. The model was not intended to represent an actual system behavior—it is simply a semi-Markov model; but, the recovery transition in this model can be interpreted to be intermittent without there being any permanent recovery transition. In this model, α represents the rate at which a fault goes from the active to the benign state and β represents the rate at which a fault goes from the benign to the active state. The fault-arrival rate is given by λ.



Example 20: Triad with intermittent faults.

The CARE III model assumes there is an additional recovery transition from state 2 to state 4 and from state 5 to state 7; hence, CARE III cannot directly model this example.

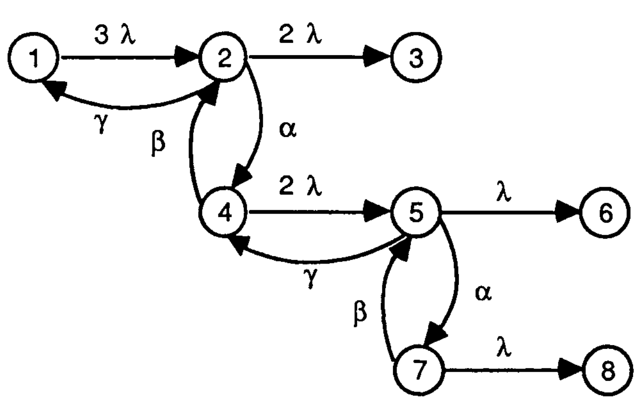
[4] CARE III's user's guide (p. D-2, ref. 8) states that separation of the values of the fault-handling parameters by more than two or three orders of magnitude may cause numerical inaccuracies. The implications of inaccuracy are not elaborated. Since some of the test cases violate this warning, one might expect CARE III's estimates to differ from the other estimates.

[5] The CARE III program uses a set of parameters called run-time parameters that indicate the operating time and the time scale for which the system is to be assessed. Some of these parameters control the amount of computation that CARE III performs (p. 30, ref. 8). The run-time parameters are not actual parameters of the model as are λ, δ, and α in example 19.

Comparison Between SURE and PAWS for Example 20			
Parameters	Death state	SURE bounds	PAWS
$\lambda = 1 \times 10^{-4}$ $\alpha = 3.6 \times 10^{4}$ $\beta = 1 \times 10^{-2}$	Total	$(1.01199 \times 10^{-9},\ 1.06907 \times 10^{-9})$	1.01597×10^{-9}
$\lambda = 1 \times 10^{-5}$ $\alpha = 3.6 \times 10^{3}$ $\beta = 1 \times 10^{-7}$	Total	$(2.63145 \times 10^{-12},\ 2.66675 \times 10^{-12})$	2.66614×10^{-12}
$\lambda = 1 \times 10^{-4}$ $\alpha = 3.6 \times 10^{3}$ $\beta = 1 \times 10^{-3}$	Total	$(1.15315 \times 10^{-9},\ 1.17260 \times 10^{-9})$	1.16566×10^{-9}
$\lambda = 1 \times 10^{-2}$ $\alpha = 3.6 \times 10^{1}$ $\beta = 1 \times 10^{-4}$	Total	$(8.23088 \times 10^{-4},\ 1.01292 \times 10^{-3})$	9.98154×10^{-4}
$\lambda = 1 \times 10^{-3}$ $\alpha = 3.6 \times 10^{6}$ $\beta = 1 \times 10^{-3}$	Total	$(9.84826 \times 10^{-7},\ 1.00503 \times 10^{-6})$	9.85141×10^{-7}
$\lambda = 1 \times 10^{-6}$ $\alpha = 3.6 \times 10^{2}$ $\beta = 1 \times 10^{-3}$	Total	$(1.60617 \times 10^{-13},\ 1.68509 \times 10^{-13})$	1.68449×10^{-13}

Example 21

Example 21 is the same triad as in the previous example except that transient faults are also considered along with the permanent and intermittent faults. The model parameters are defined as follows: λ is the fault-arrival rate, α is the rate at which an intermittent fault goes from the active to the benign state, β is the rate at which an intermittent fault goes from the benign to the active state, and γ is the transient-fault-disappearance rate.



Example 21: Triad with permanent, intermittent, and transient faults.

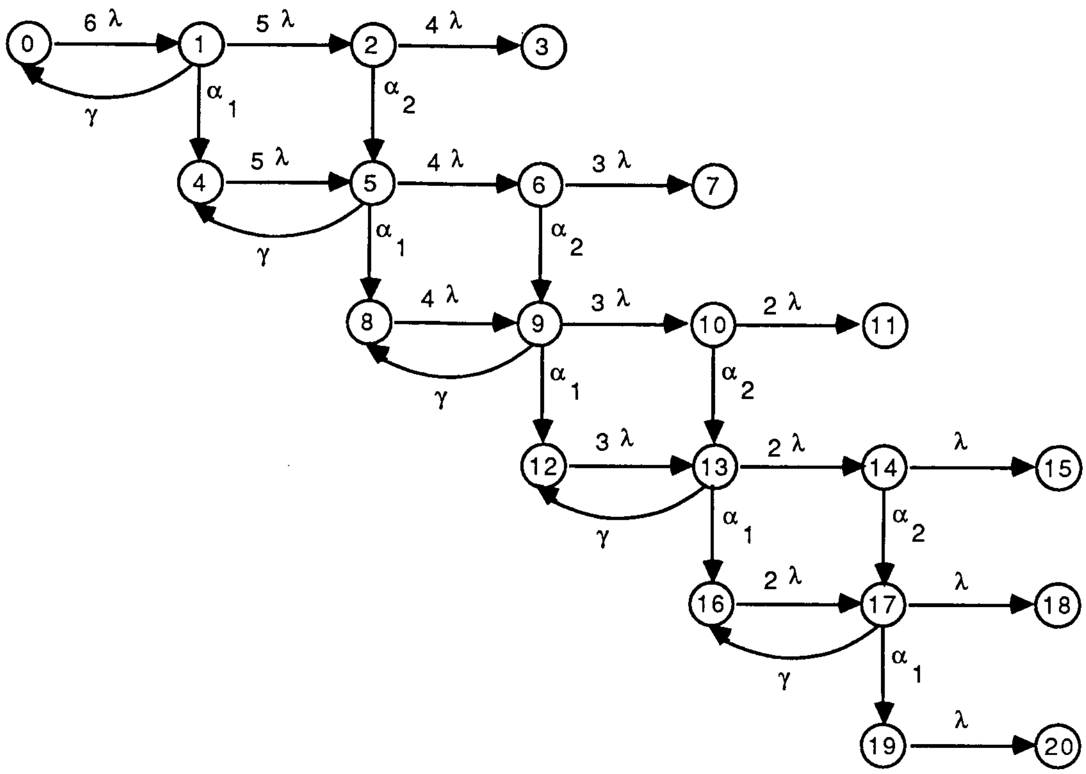
As in the previous example, the CARE III model assumes there are additional recovery transitions; hence, CARE III cannot directly model this example.

Comparison Between SURE and PAWS for Example 21			
Parameters	Death state	SURE bounds	PAWS
$\lambda = 1 \times 10^{-4}$ $\alpha = 3.6 \times 10^{3}$ $\beta = 1 \times 10^{-2}$ $\gamma = 3.6 \times 10^{1}$	Total	$(1.13695 \times 10^{-9},\ 1.20386 \times 10^{-9})$	1.15119×10^{-9}
$\lambda = 1 \times 10^{-6}$ $\alpha = 3.6 \times 10^{6}$ $\beta = 1 \times 10^{-5}$ $\gamma = 3.6 \times 10^{4}$	Total	$(9.96675 \times 10^{-16},\ 9.96846 \times 10^{-16})$	9.96783×10^{-16}
$\lambda = 1 \times 10^{-6}$ $\alpha = 3.6 \times 10^{2}$ $\beta = 1 \times 10^{-7}$ $\gamma = 3.6 \times 10^{1}$	Total	$(1.45571 \times 10^{-13},\ 1.52343 \times 10^{-13})$	1.52301×10^{-13}
$\lambda = 1 \times 10^{-3}$ $\alpha = 3.6 \times 10^{5}$ $\beta = 1 \times 10^{-5}$ $\gamma = 3.6 \times 10^{2}$	Total	$(9.82746 \times 10^{-7},\ 9.98233 \times 10^{-7})$	9.83332×10^{-7}
$\lambda = 1 \times 10^{-4}$ $\alpha = 3.6 \times 10^{4}$ $\beta = 1 \times 10^{-1}$ $\gamma = 1 \times 10^{4}$	Total	$(5.65847 \times 10^{-10},\ 5.67191 \times 10^{-10})$	5.67165×10^{-10}
$\lambda = 1 \times 10^{-2}$ $\alpha = 3.6 \times 10^{6}$ $\beta = 1 \times 10^{-1}$ $\gamma = 3.6 \times 10^{5}$	Total	$(6.89651 \times 10^{-4},\ 6.89773 \times 10^{-4})$	6.89742×10^{-4}

Example 22

Example 22 is a system of six components that are vulnerable to both permanent and transient faults. Critical triples, where three nearly simultaneous faults can cause system failure, are also considered in this model. The parameters for the model are defined as follows: λ is the fault-arrival rate, α_1 and α_2 are the recovery rates from the first and second faults, respectively, and γ is the transient-fault-disappearance rate.

As mentioned earlier, CARE III cannot directly model this system because of the presence of critical-triple faults. As argued before, one might expect that CARE III would give a conservative estimate of unreliability in cases with critical triples. CARE III's best estimate for the model is given in the following table. The CARE III estimates in many of these test cases were actually unconservative.

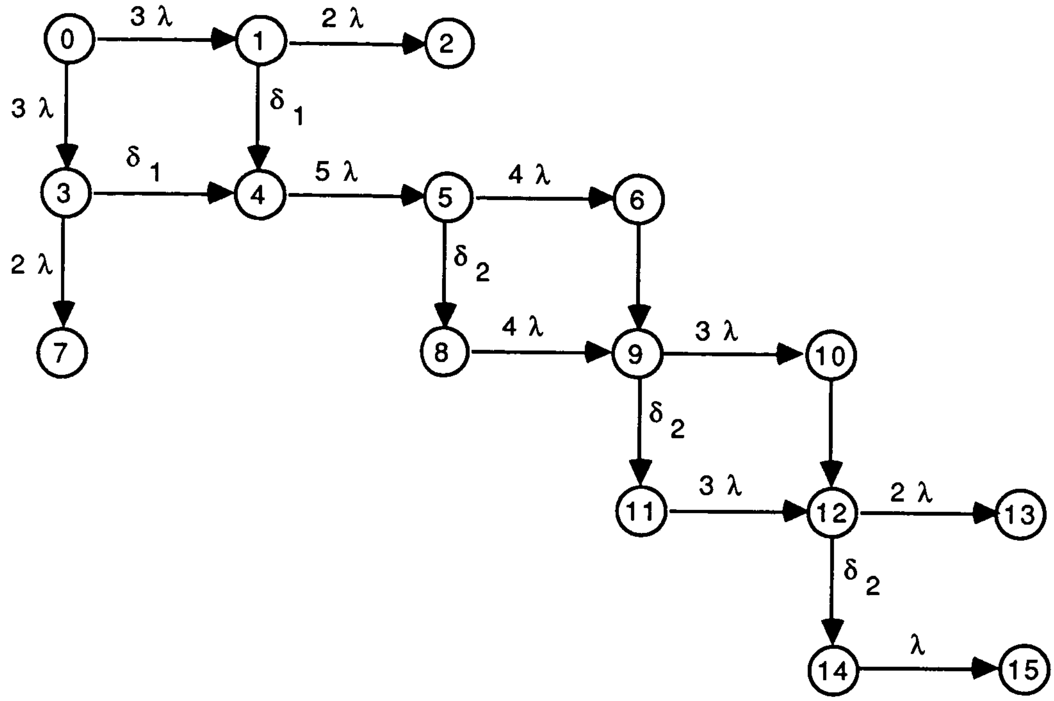


Example 22: Critical-triple six-plex with transient faults.

Comparison Among SURE, PAWS, and CARE III for Example 22				
Parameters	Death state	SURE bounds	PAWS	CARE III
$\lambda = 1 \times 10^{-2}$ $\alpha_1 = 1 \times 10^{6}$ $\alpha_2 = 1 \times 10^{6}$ $\gamma = 1$	Total	$(7.41883 \times 10^{-7},\ 7.42671 \times 10^{-7})$	7.42669×10^{-7}	7.57083×10^{-7}
$\lambda = 1 \times 10^{-6}$ $\alpha_1 = 1 \times 10^{2}$ $\alpha_2 = 1 \times 10^{6}$ $\gamma = 1 \times 10^{2}$	Total	$(5.62137 \times 10^{-24},\ 6.00014 \times 10^{-24})$	5.99695×10^{-24}	2.53437×10^{-23}
$\lambda = 1 \times 10^{-3}$ $\alpha_1 = 1 \times 10^{5}$ $\alpha_2 = 1 \times 10^{4}$ $\gamma = 1 \times 10^{2}$	Total	$(9.61445 \times 10^{-13},\ 9.96290 \times 10^{-13})$	9.66841×10^{-13}	2.82760×10^{-12}
$\lambda = 1 \times 10^{-3}$ $\alpha_1 = 1 \times 10^{6}$ $\alpha_2 = 1 \times 10^{3}$ $\gamma = 1 \times 10^{2}$	Total	$(9.69596 \times 10^{-13},\ 1.00072 \times 10^{-12})$	9.71169×10^{-13}	1.53586×10^{-11}
$\lambda = 1 \times 10^{-6}$ $\alpha_1 = 1 \times 10^{4}$ $\alpha_2 = 1 \times 10^{4}$ $\gamma = 1 \times 10^{2}$	Total	$(1.16702 \times 10^{-23},\ 1.18814 \times 10^{-23})$	1.18808×10^{-23}	1.95616×10^{-24}
$\lambda = 1 \times 10^{-4}$ $\alpha_1 = 1 \times 10^{1}$ $\alpha_2 = 1 \times 10^{5}$ $\gamma = 1 \times 10^{5}$	Total	$(1.19360 \times 10^{-19},\ 1.20349 \times 10^{-19})$	1.19988×10^{-19}	0.00000
$\lambda = 1 \times 10^{-5}$ $\alpha_1 = 1 \times 10^{4}$ $\alpha_2 = 1$ $\gamma = 1 \times 10^{4}$	Total	$(1.81698 \times 10^{-17},\ 6.00135 \times 10^{-17})$*	5.39944×10^{-17}	0.00000
*DELTA > TIME				

Example 23

Example 23 consists of two triads of components. When there is a fault in either of the triads, the system reconfigures with rate δ_1 into a single system of five components. Only critical-pair failures or exhaustion of components result in failure of the system. The failure rate of any component is λ, and δ_2 represents the recovery rate once the two triads have formed a five-plex. CARE III cannot model this example since the behavior of this system cannot be captured in a fault-tree representation.

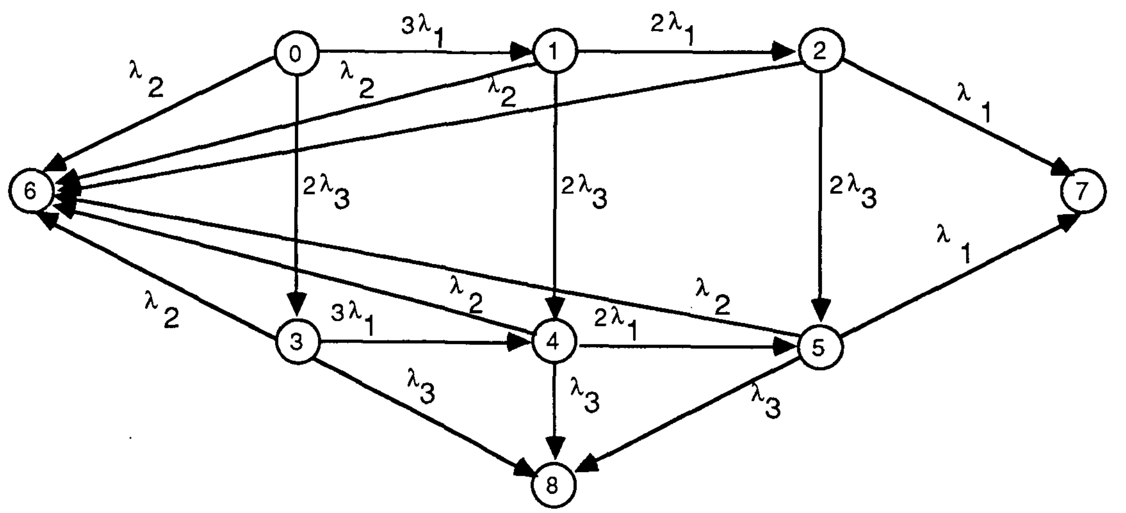


Example 23: Two triads with permanent faults.

Comparison Between SURE and PAWS for Example 23			
Parameters	Death state	SURE bounds	PAWS
$\lambda = 1 \times 10^{-4}$ $\delta_1 = 1 \times 10^5$ $\delta_2 = 1 \times 10^5$	Total	$(1.19928 \times 10^{-11},\ 1.20631 \times 10^{-11})$	1.20269×10^{-11}
$\lambda = 1 \times 10^{-5}$ $\delta_1 = 1 \times 10^5$ $\delta_2 = 1 \times 10^1$	Total	$(4.99102 \times 10^{-13},\ 7.20060 \times 10^{-13})$	7.07923×10^{-13}
$\lambda = 1 \times 10^{-4}$ $\delta_1 = 1 \times 10^2$ $\delta_2 = 1 \times 10^4$	Total	$(1.09199 \times 10^{-9},\ 1.20006 \times 10^{-8})$	1.19527×10^{-8}
$\lambda = 1 \times 10^{-7}$ $\delta_1 = 1 \times 10^3$ $\delta_2 = 1 \times 10^4$	Total	$(1.16630 \times 10^{-15},\ 1.20000 \times 10^{-15})$	1.19988×10^{-15}
$\lambda = 1 \times 10^{-6}$ $\delta_1 = 1 \times 10^5$ $\delta_2 = 1 \times 10^4$	Total	$(1.19716 \times 10^{-15},\ 1.20060 \times 10^{-15})$	1.20056×10^{-15}
$\lambda = 1 \times 10^{-2}$ $\delta_1 = 1 \times 10^4$ $\delta_2 = 1 \times 10^4$	Total	$(2.29458 \times 10^{-5},\ 2.32928 \times 10^{-5})$	2.32884×10^{-5}
$\lambda = 1 \times 10^{-3}$ $\delta_1 = 1 \times 10^2$ $\delta_2 = 1 \times 10^1$	Total	$(1.41745 \times 10^{-6},\ 1.80633 \times 10^{-6})$	1.73510×10^{-6}
$\lambda = 1 \times 10^{-4}$ $\delta_1 = 1 \times 10^6$ $\delta_2 = 1 \times 10^2$	Total	$(5.33614 \times 10^{-11},\ 6.12630 \times 10^{-11})$	6.09198×10^{-11}

Example 24

A system of three processors, two memories, and one bus is modeled in example 24. Exhaustion of any one of the three component types causes the system to fail. In this model, λ_1 is the failure rate of a processor, λ_2 is the failure rate of a memory, and λ_3 is the failure rate of a bus. System failure due to bus failure is represented by state 6; system failure due to exhaustion of processors is represented by state 7; and system failure due to exhaustion of memories is represented by state 8.

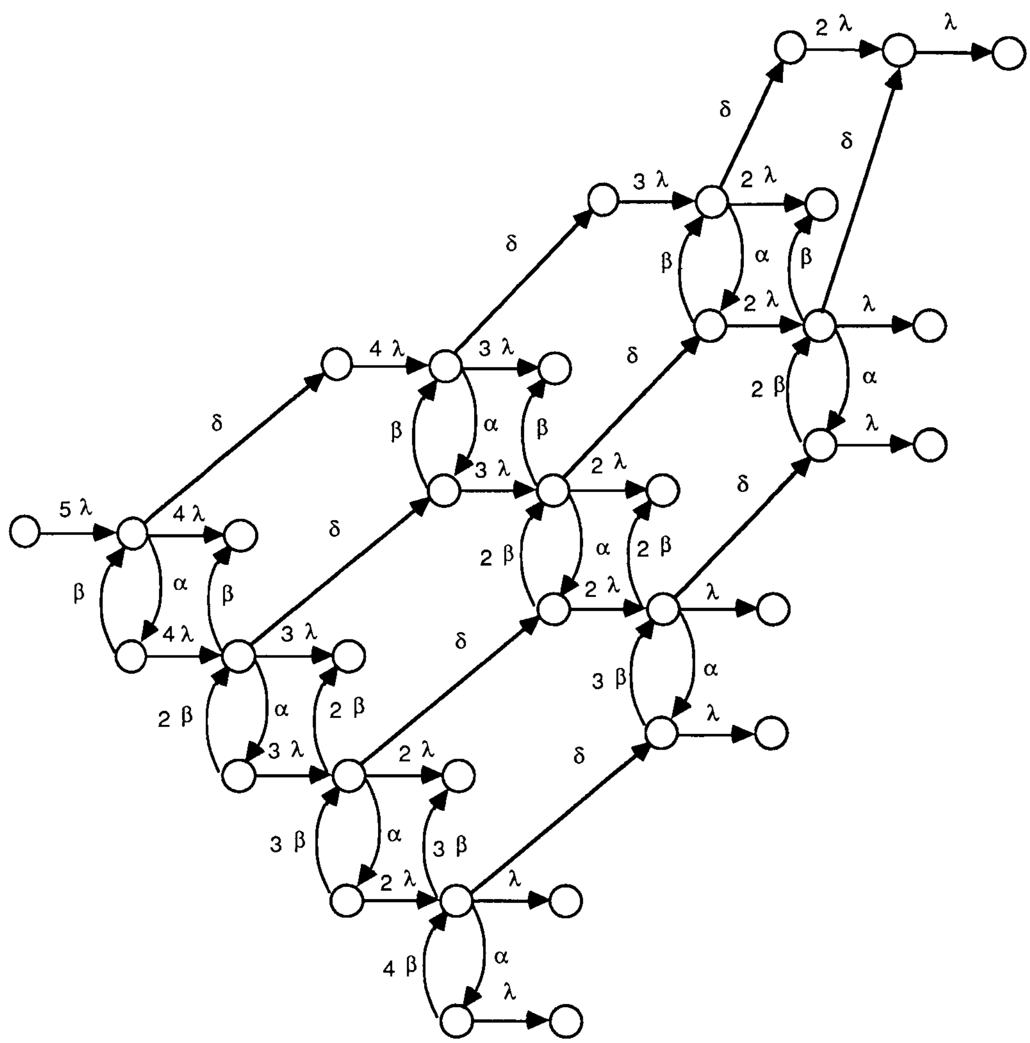


Example 24: System of three processors, two memories, and one bus.

Comparison Among SURE, PAWS, and CARE III for Example 24				
Parameters	Death state	SURE bounds	PAWS	CARE III
$\lambda_1 = 1 \times 10^{-3}$	$D_6(T)$	$(9.94373 \times 10^{-3},\ 1.02530 \times 10^{-2})$	9.94984×10^{-3}	
$\lambda_2 = 1 \times 10^{-3}$	$D_7(T)$	$(9.76960 \times 10^{-7},\ 1.01500 \times 10^{-6})$	9.77715×10^{-7}	
$\lambda_3 = 1 \times 10^{-3}$	$D_8(T)$	$(9.82728 \times 10^{-5},\ 1.02015 \times 10^{-4})$	9.83490×10^{-5}	
	Total	$(1.00430 \times 10^{-2},\ 1.03560 \times 10^{-2})$	1.00492×10^{-2}	1.00492×10^{-2}
$\lambda_1 = 1 \times 10^{-7}$	$D_6(T)$	$(9.99761 \times 10^{-1},\ 9.99761 \times 10^{-1})$	9.99761×10^{-1}	
$\lambda_2 = 1 \times 10^{-2}$	$D_7(T)$	$(5.97672 \times 10^{-21},\ 5.97672 \times 10^{-21})$	5.97672×10^{-21}	
$\lambda_3 = 1$	$D_8(T)$	$(1.94052 \times 10^{-4},\ 1.94052 \times 10^{-4})$	1.94052×10^{-4}	
	Total	$(9.99955 \times 10^{-1},\ 9.99955 \times 10^{-1})$	9.99955×10^{-1}	9.99955×10^{-1}
$\lambda_1 = 1 \times 10^{-3}$	$D_6(T)$	$(9.94748 \times 10^{-3},\ 1.01510 \times 10^{-2})$	9.95016×10^{-3}	
$\lambda_2 = 1 \times 10^{-7}$	$D_7(T)$	$(9.77500 \times 10^{-7},\ 1.00000 \times 10^{-6})$	9.77773×10^{-7}	
$\lambda_3 = 1 \times 10^{-3}$	$D_8(T)$	$(9.92954 \times 10^{-13},\ 1.02015 \times 10^{-12})$	9.93357×10^{-13}	
	Total	$(9.94845 \times 10^{-3},\ 1.01520 \times 10^{-2})$	9.95114×10^{-3}	9.95114×10^{-3}
$\lambda_1 = 1$	$D_6(T)$	$(1.80937 \times 10^{-2},\ 1.80937 \times 10^{-2})$	1.80937×10^{-2}	
$\lambda_2 = 1 \times 10^{-2}$	$D_7(T)$	$(9.81338 \times 10^{-1},\ 9.81338 \times 10^{-1})$	9.81338×10^{-1}	
$\lambda_3 = 1 \times 10^{-2}$	$D_8(T)$	$(4.46411 \times 10^{-4},\ 4.46411 \times 10^{-4})$	4.46411×10^{-4}	
	Total	$(9.99878 \times 10^{-1},\ 9.99878 \times 10^{-1})$	9.99878×10^{-1}	9.99878×10^{-1}
$\lambda_1 = 1 \times 10^{-6}$	$D_6(T)$	$(9.99498 \times 10^{-4},\ 1.00102 \times 10^{-3})$	9.99500×10^{-4}	
$\lambda_2 = 1 \times 10^{-4}$	$D_7(T)$	$(9.99232 \times 10^{-16},\ 1.00150 \times 10^{-15})$	9.99235×10^{-16}	
$\lambda_3 = 1 \times 10^{-4}$	$D_8(T)$	$(9.98333 \times 10^{-7},\ 1.00002 \times 10^{-6})$	9.98335×10^{-7}	
	Total	$(1.00050 \times 10^{-3},\ 1.00202 \times 10^{-3})$	1.00050×10^{-3}	1.00050×10^{-3}
$\lambda_1 = 1 \times 10^{-4}$	$D_6(T)$	$(9.96900 \times 10^{-6},\ 9.96900 \times 10^{-6})$	9.96900×10^{-6}	
$\lambda_2 = 1 \times 10^{-2}$	$D_7(T)$	$(9.92979 \times 10^{-10},\ 9.92979 \times 10^{-10})$	9.92979×10^{-10}	
$\lambda_3 = 1 \times 10^{-6}$	$D_8(T)$	$(9.05586 \times 10^{-3},\ 9.05586 \times 10^{-3})$	9.05586×10^{-3}	
	Total	$(9.06583 \times 10^{-3},\ 9.06583 \times 10^{-3})$	9.06583×10^{-3}	9.06582×10^{-3}

Example 25

Example 25 consists of a system of five components that are critically paired and are susceptible to intermittent faults. The parameters of the model are defined as follows: λ is the fault-arrival rate, α is the rate at which an intermittent fault goes from the active to the benign state, β is the rate at which an intermittent fault goes from the benign to the active state, and δ is the permanent-fault-recovery rate.



Example 25: Critical-pair five-plex with intermittent faults.

Comparison Among SURE, PAWS, and CARE III for Example 25				
	Total system unreliability			
Parameters	SURE bounds	PAWS	CARE III	CARE III*
$\lambda = 1 \times 10^{-6}$ $\alpha = 3.6 \times 10^{-3}$ $\beta = 1 \times 10^{-2}$ $\delta = 3.6 \times 10^{6}$	$(1.38822 \times 10^{-17},\ 1.38889 \times 10^{-17})$	1.38888×10^{-17}	5.57649×10^{-17}	5.57658×10^{-17}
$\lambda = 1 \times 10^{-6}$ $\alpha = 3.6 \times 10^{-2}$ $\beta = 1 \times 10^{1}$ $\delta = 3.6 \times 10^{2}$	$(5.29698 \times 10^{-13},\ 5.55674 \times 10^{-13})$	5.55449×10^{-13}	5.57028×10^{-13}	5.57036×10^{-13}
$\lambda = 1 \times 10^{-6}$ $\alpha = 3.6 \times 10^{-1}$ $\beta = 1 \times 10^{-1}$ $\delta = 3.6 \times 10^{2}$	$(5.29530 \times 10^{-13},\ 5.55417 \times 10^{-13})$	5.55249×10^{-13}	5.53959×10^{-13}	5.53967×10^{-13}
$\lambda = 2 \times 10^{-7}$ $\alpha = 3.6 \times 10^{-3}$ $\beta = 3.6 \times 10^{-2}$ $\delta = 3.6 \times 10^{1}$	$(1.90313 \times 10^{-13},\ 2.22230 \times 10^{-13})$	2.21589×10^{-13}	3.19998×10^{-29}	2.21615×10^{-13}
$\lambda = 1 \times 10^{-7}$ $\alpha = 3.6$ $\beta = 1 \times 10^{-1}$ $\delta = 3.6 \times 10^{2}$	$(5.28272 \times 10^{-15},\ 5.54082 \times 10^{-15})$	5.53926×10^{-15}	9.99998×10^{-31}	5.50788×10^{-15}
$\lambda = 1 \times 10^{-8}$ $\alpha = 1 \times 10^{-1}$ $\beta = 1 \times 10^{2}$ $\delta = 1 \times 10^{6}$	$(1.99821 \times 10^{-20},\ 2.00000 \times 10^{-20})$	2.00000×10^{-20}	3.12500×10^{-32}	2.00275×10^{-20}
$\lambda = 1 \times 10^{-4}$ $\alpha = 1 \times 10^{2}$ $\beta = 1 \times 10^{1}$ $\delta = 1 \times 10^{3}$	$(2.13068 \times 10^{-9},\ 2.19804 \times 10^{-9})$	2.19131×10^{-9}	2.18797×10^{-9}	2.18797×10^{-9}
*PSTRNC reduced to 10^{-30}				

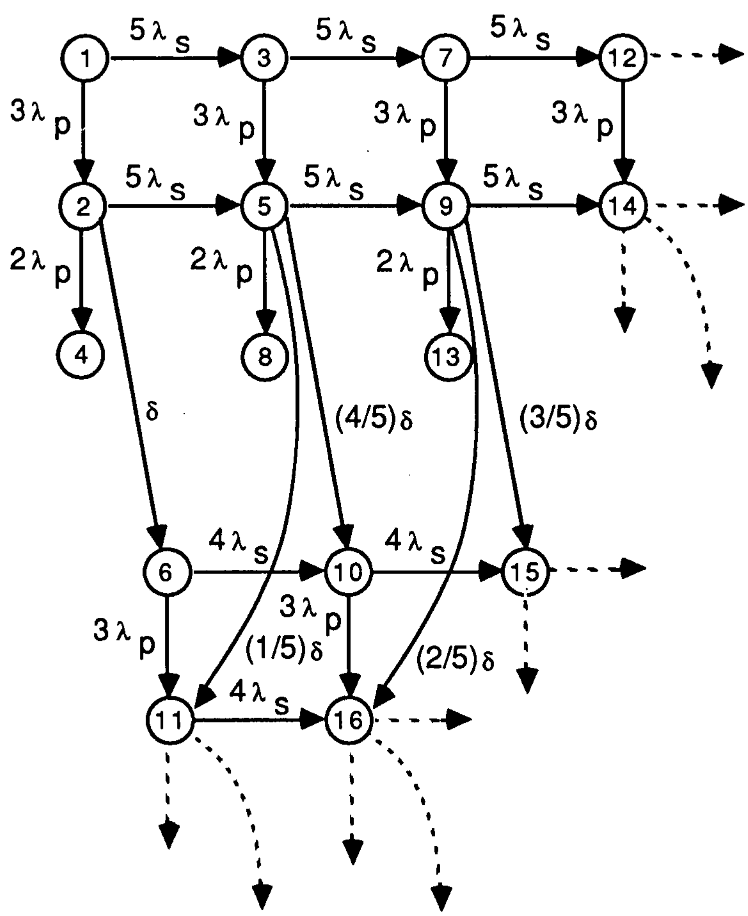
The results listed under the CARE III* column were obtained by reducing one of the run-time parameters, PSTRNC,[6] of the CARE III program from the default value of 10^{-10} to 10^{-30}. When the above test cases were run with CARE III with all the default run-time parameters, many of the unreliability estimates (shown in the first CARE III column) were off by several orders of magnitude. More importantly, these estimates were not conservative and no warning messages were output by CARE III. After experimenting with several of the run-time parameters, reducing the PSTRNC parameter helped CARE III to give better estimates. Since the difference between the estimates in the CARE III* column and the other estimates

[6] The PSTRNC parameter is used to limit the number of fault vectors that CARE III uses in computing the fault-handling unreliability. Only the fault vectors whose module depletion probability is less than PSTRNC are included in the fault-handling unreliability calculation (p. 31, ref. 8).

is relatively small, the CARE III* estimates may be considered good enough for use based on engineering judgment. However, some of the estimates given by CARE III are still not conservative, and the source of this error is not evident.

Example 26

The system modeled in example 26 is a triad of processors with five cold spare processors. The system is operational as long as there are two working processors. Replacement of a failed processor with a failed spare is possible. Since the model is large, 63 states and 102 transitions, only part of the model is shown in the sketch. The construction of the entire model should be obvious, though, from the portion shown and the model description. The parameters of the model are as follows: λ_p is the failure rate of an active processor, λ_s is the failure rate of a cold spare, and δ is the fault-recovery rate. This system cannot be modeled with CARE III since the spares are cold. CARE III assumes that a spare module has the identical fault-handling characteristics as that module does when it is being used by the system. If $\lambda_p = \lambda_s$, CARE III could be used for this example.

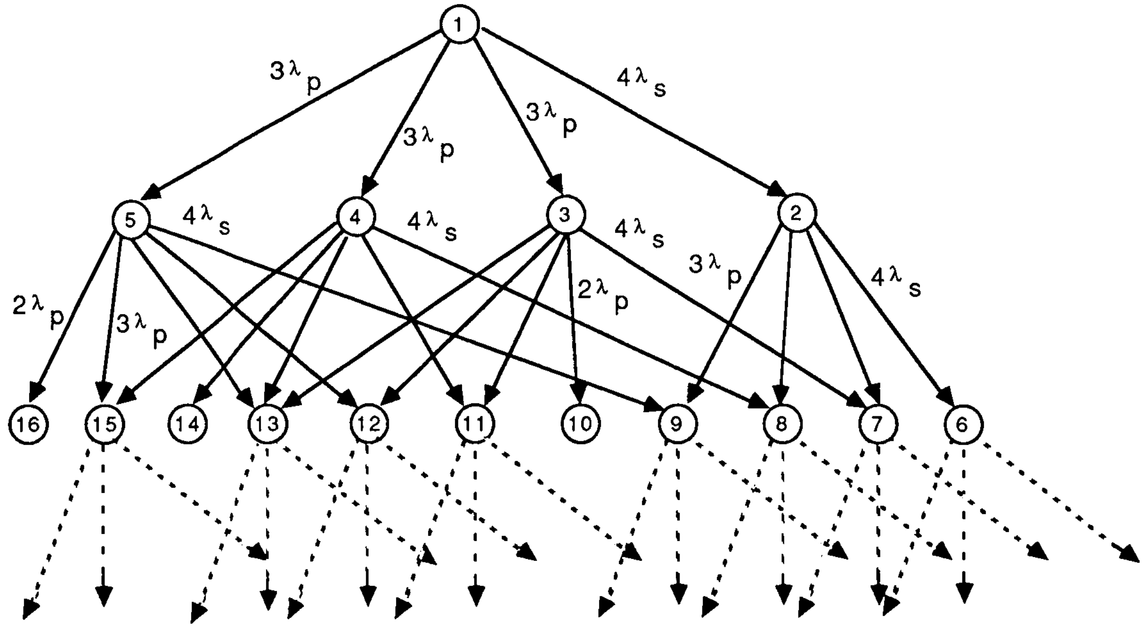


Example 26: Triad with five cold spares.

Comparison Between SURE and PAWS for Example 26			
Parameters	Death state	SURE bounds	PAWS
$\lambda_p = 1 \times 10^{-4}$ $\lambda_s = 1 \times 10^{-5}$ $\delta = 3.6 \times 10^{3}$	Total	$(1.64198 \times 10^{-10},\ 1.66967 \times 10^{-10})$	1.66670×10^{-10}
$\lambda_p = 1 \times 10^{-6}$ $\lambda_s = 1 \times 10^{-3}$ $\delta = 3.6 \times 10^{3}$	Total	$(1.64941 \times 10^{-14},\ 1.71776 \times 10^{-14})$	1.67503×10^{-14}
$\lambda_p = 1 \times 10^{-3}$ $\lambda_s = 1 \times 10^{-6}$ $\delta = 3.6 \times 10^{3}$	Total	$(1.64159 \times 10^{-8},\ 1.69197 \times 10^{-8})$	1.66663×10^{-8}
$\lambda_p = 1 \times 10^{-4}$ $\lambda_s = 1 \times 10^{-4}$ $\delta = 3.6 \times 10^{1}$	Total	$(1.42790 \times 10^{-8},\ 1.67422 \times 10^{-8})$	1.66284×10^{-8}
$\lambda_p = 1 \times 10^{-2}$ $\lambda_s = 1 \times 10^{-5}$ $\delta = 3.6 \times 10^{5}$	Total	$(3.92378 \times 10^{-8},\ 3.93185 \times 10^{-8})$	3.93183×10^{-8}
$\lambda_p = 1 \times 10^{-2}$ $\lambda_s = 1 \times 10^{-4}$ $\delta = 3.6 \times 10^{2}$	Total	$(1.58754 \times 10^{-5},\ 1.66986 \times 10^{-5})$	1.66913×10^{-5}
$\lambda_p = 1 \times 10^{-6}$ $\lambda_s = 1 \times 10^{-5}$ $\delta = 3.6 \times 10^{4}$	Total	$(1.65890 \times 10^{-15},\ 1.66719 \times 10^{-15})$	1.66674×10^{-15}

Example 27

The model for example 27 represents a system of three triads of processors with a pool of four spare processors. Two faults in an active triad cause system failure, and the system also fails if there is only one working triad. A failed triad is also broken up when no spares are available. This model has a total of 450 states and 946 transitions. Since this model contains more than 300 states, PAWS cannot be used. The program STEM was used for this example since it does have the capability to compute the exact death-state probabilities for Markov models as large as 1000 states. The parameters of the model are defined as follows: λ_p is the failure rate of an active processor, λ_s is the failure rate of a cold spare, δ_1 is the reconfiguration rate to switch in a spare, and δ_2 is the reconfiguration rate to break up a triad.

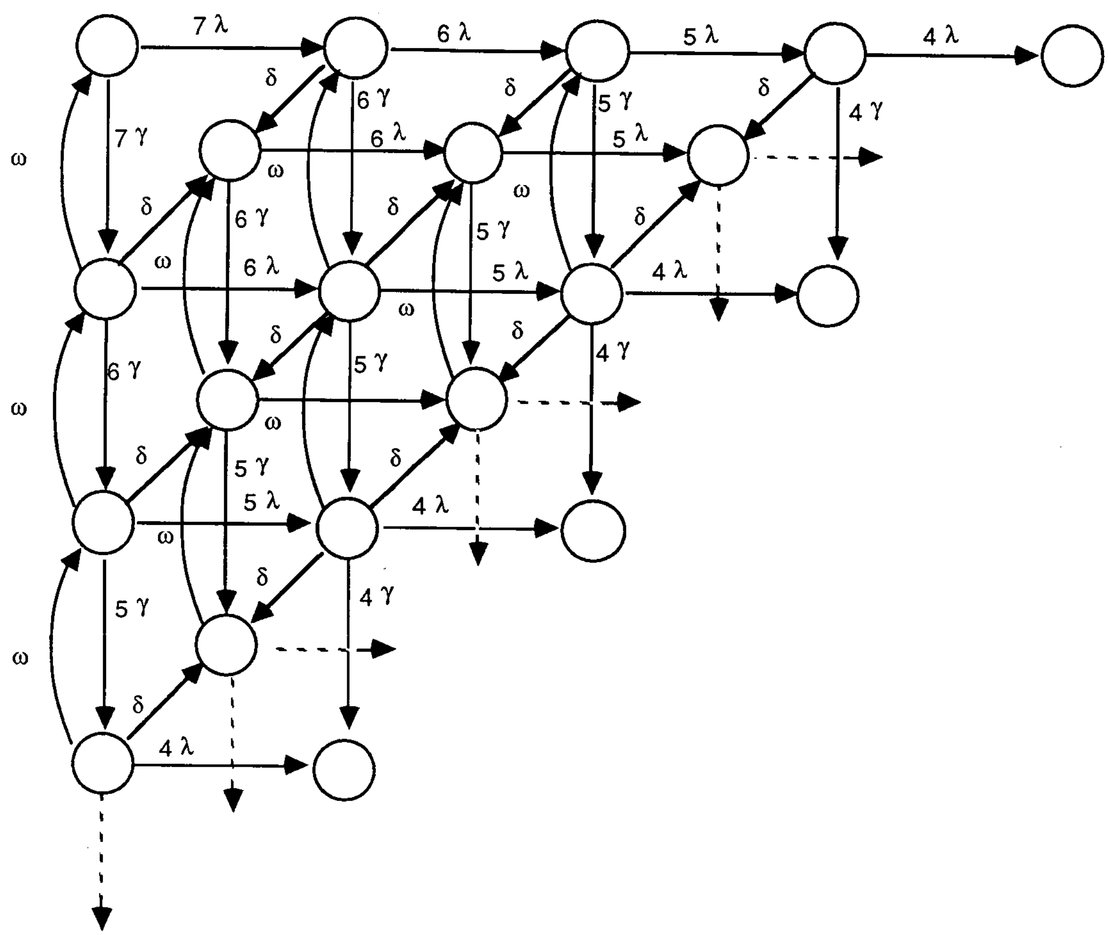


Example 27: System of three triads with a pool of four spares.

Comparison Between SURE and STEM for Example 27			
Parameters	Death state	SURE bounds	STEM
$\lambda_p = 1 \times 10^{-4}$ $\lambda_s = 1 \times 10^{-5}$ $\delta_1 = 3.6 \times 10^3$ $\delta_2 = 5.1 \times 10^3$	Total	$(4.92568 \times 10^{-10},\ 5.02425 \times 10^{-10})$	5.00011×10^{-10}
$\lambda_p = 1 \times 10^{-6}$ $\lambda_s = 1 \times 10^{-3}$ $\delta_1 = 3.6 \times 10^3$ $\delta_2 = 3.6 \times 10^3$	Total	$(4.93221 \times 10^{-14},\ 5.17200 \times 10^{-14})$	5.02509×10^{-14}
$\lambda_p = 1 \times 10^{-2}$ $\lambda_s = 1 \times 10^{-4}$ $\delta_1 = 3.6 \times 10^4$ $\delta_2 = 3.6 \times 10^1$	Total	$(2.32096 \times 10^{-6},\ 3.45050 \times 10^{-6})$	3.36494×10^{-6}
$\lambda_p = 1 \times 10^{-4}$ $\lambda_s = 1 \times 10^{-5}$ $\delta_1 = 3.6 \times 10^2$ $\delta_2 = 3.6 \times 10^5$	Total	$(4.76688 \times 10^{-9},\ 5.02384 \times 10^{-9})$	4.99885×10^{-9}
$\lambda_p = 1 \times 10^{-3}$ $\lambda_s = 1 \times 10^{-4}$ $\delta_1 = 3.6 \times 10^5$ $\delta_2 = 3.6 \times 10^2$	Total	$(4.98999 \times 10^{-10},\ 5.24846 \times 10^{-10})$	5.00567×10^{-10}

Example 28

Example 28 is a system of seven components that are susceptible to transient faults. The system fails if a majority of the active components have failed. The model has the following parameters: λ is the permanent-fault-arrival rate, γ is the transient-fault-arrival rate, ω is the transient-fault-disappearance rate, and δ is the fault-recovery rate. Since the model is large, 50 states and 100 transitions, only a portion of the model is shown in the figure.



Example 28: Seven-plex with transient faults.

Comparison Between SURE and PAWS for Example 28			
Parameters	Death state	SURE bounds	PAWS
$\lambda = 1 \times 10^{-4}$ $\gamma = 1 \times 10^{-3}$ $\omega = 5 \times 10^{-1}$ $\delta = 3.6 \times 10^{3}$	Total	$(1.17543 \times 10^{-11},\ 1.28755 \times 10^{-11})$	1.23283×10^{-11}
$\lambda = 1 \times 10^{-5}$ $\gamma = 1 \times 10^{-2}$ $\omega = 1 \times 10^{-3}$ $\delta = 3.6 \times 10^{3}$	Total	$(4.61234 \times 10^{-6},\ 4.82377 \times 10^{-6})$	4.81975×10^{-6}
$\lambda = 1 \times 10^{-3}$ $\gamma = 1 \times 10^{-3}$ $\omega = 1 \times 10^{-2}$ $\delta = 3.6 \times 10^{5}$	Total	$(4.12632 \times 10^{-10},\ 4.48096 \times 10^{-10})$	4.14876×10^{-10}
$\lambda = 1 \times 10^{-3}$ $\gamma = 1 \times 10^{-4}$ $\omega = 5 \times 10^{-1}$ $\delta = 3.6 \times 10^{2}$	Total	$(1.35520 \times 10^{-11},\ 1.76273 \times 10^{-11})$	1.67470×10^{-11}
$\lambda = 1 \times 10^{-4}$ $\gamma = 1 \times 10^{-6}$ $\omega = 1 \times 10^{-2}$ $\delta = 3.6 \times 10^{4}$	Total	$(7.63071 \times 10^{-18},\ 7.74103 \times 10^{-18})$	7.71014×10^{-18}
$\lambda = 1 \times 10^{-2}$ $\gamma = 1 \times 10^{-4}$ $\omega = 1 \times 10^{-3}$ $\delta = 3.6 \times 10^{2}$	Total	$(4.27984 \times 10^{-6},\ 5.28384 \times 10^{-6})$	5.24063×10^{-6}

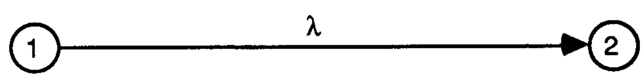
Appendix C

Additional Comparisons of SURE With CARE III and ARIES

Recall that the following models, examples 29 through 35, were taken from reference 14. This report describes a comparative analysis between the ARIES and CARE III tools designed to show the strengths and weaknesses of each tool for analyzing architectures for fault-tolerant aerospace systems. Seven simple reliability models were analyzed with ARIES and CARE III and compared with a direct calculation of the unreliability of the modeled system. The SURE program was run for each of these models, and SURE's bounds are given along with the corresponding results from ARIES, CARE III, and a direct calculation of the unreliability.

Example 29

Example 29 represents the most basic construction in any Markov model—the failure of a solitary component such as a processor. The model is a simple two-state model where the component failure rate is λ.



Example 29: Failure of a single component.

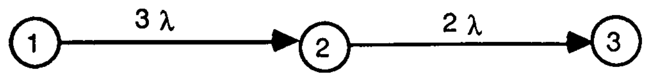
The analytic solution for the probability of the failure of this system during a mission time T is $D_2(T) = 1 - e^{-\lambda T}$. When λT is very small $D_2(T) \approx \lambda T$.

Comparison of SURE, ARIES, and CARE III for Example 29				
Parameter λT	Analytic solution	ARIES	CARE III	SURE bounds
0	0	0	0	(0.00000, 0.00000)
1.59×10^{-23}	1.59000×10^{-23}	0	1.59×10^{-23}	$(1.59000 \times 10^{-23},\ 1.59000 \times 10^{-23})$
1.30×10^{-19}	1.30000×10^{-19}	0	1.30×10^{-19}	$(1.30000 \times 10^{-19},\ 1.30000 \times 10^{-19})$
1.0×10^{-16}	1.00000×10^{-16}	6.94×10^{-17}	9.99×10^{-17}	$(1.00000 \times 10^{-16},\ 1.00000 \times 10^{-16})$
5.0×10^{-16}	5.00000×10^{-16}	4.85×10^{-16}	5.00×10^{-16}	$(5.00000 \times 10^{-16},\ 5.00000 \times 10^{-16})$
1.0×10^{-15}	1.00000×10^{-15}	9.99×10^{-16}	1.00×10^{-15}	$(1.00000 \times 10^{-15},\ 1.00000 \times 10^{-15})$
1.0×10^{-12}	1.00000×10^{-12}	9.99×10^{-13}	9.99×10^{-13}	$(1.00000 \times 10^{-12},\ 1.00000 \times 10^{-12})$
1.0×10^{-10}	1.00000×10^{-10}	9.99×10^{-11}	1.00×10^{-10}	$(1.00000 \times 10^{-10},\ 1.00000 \times 10^{-10})$
1.0×10^{-3}	9.99500×10^{-4}	9.995×10^{-4}	9.995×10^{-4}	$(9.99500 \times 10^{-4},\ 0.00000 \times 10^{-3})$
1.0	6.32121×10^{-1}	6.32×10^{-1}	6.32×10^{-1}	$(6.32121 \times 10^{-1},\ 6.32121 \times 10^{-1})$

Example 30

In example 30, a triad of components is represented. This system simply degrades with no recovery mechanism, and system failure occurs when there is only one working component. The fault-arrival rate is λ. For all the test cases for this model, $\lambda = 10^{-4}$.

For this example, the analytic solution for the death-state probability for a given mission time T is $D_3(T) = 1 - 2e^{-3\lambda T} - 3e^{-2\lambda T}$.

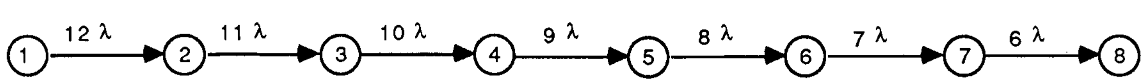


Example 30: Triad with no recovery and with no spares.

Comparison of SURE, ARIES, and CARE III for Example 30				
Parameter T	Analytic solution	ARIES	CARE III	SURE bounds
0	0	0	0	(0.00000, 0.00000)
1	2.99950×10^{-8}	2.99950×10^{-8}	2.99950×10^{-8}	$(2.99950 \times 10^{-8},\ 3.00000 \times 10^{-8})$
5	7.49375×10^{-7}	7.49375×10^{-7}	7.49374×10^{-7}	$(7.49375 \times 10^{-7},\ 7.50000 \times 10^{-7})$
10	2.99500×10^{-6}	2.99500×10^{-6}	2.99500×10^{-6}	$(2.99500 \times 10^{-6},\ 3.00000 \times 10^{-6})$
.01	3.00000×10^{-12}	2.99996×10^{-12}	2.99999×10^{-12}	$(3.00000 \times 10^{-12},\ 3.00000 \times 10^{-12})$
.10	2.99995×10^{-10}	2.99999×10^{-10}	2.99995×10^{-10}	$(2.99999 \times 10^{-10},\ 3.00000 \times 10^{-10})$
7000	5.05122×10^{-1}	5.05122×10^{-1}	5.05122×10^{-1}	$(5.05122 \times 10^{-1},\ 5.05122 \times 10^{-1})$

Example 31

Example 31 is a system of 12 components that fail with rate λ. As in the previous model, there is no recovery process and there are no spares.



Example 31: Twelve-plex with no recovery and no spares.

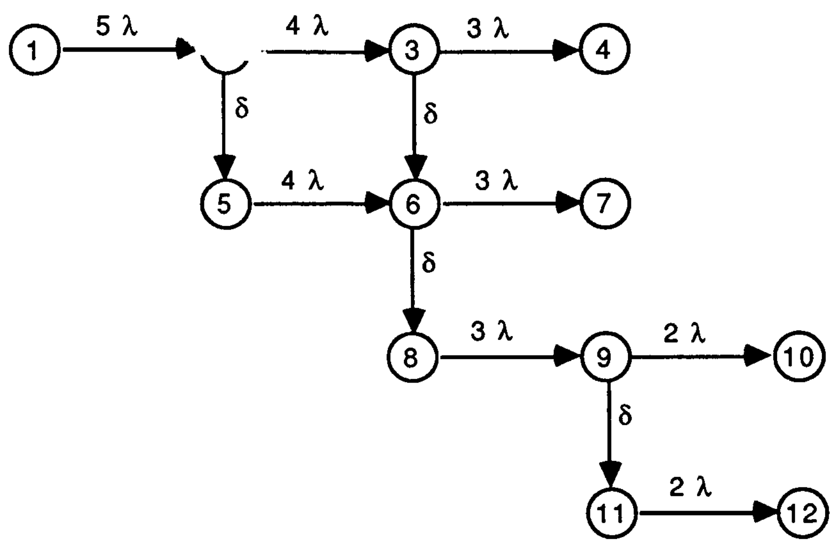
For this model, the analytic solution for the unreliability of the system D_8 for a given mission time T is

$$D_8(T) = \sum_{k=7}^{12} \frac{12!(1 - e^{-\lambda T})^k (e^{-\lambda T})^{12-k}}{k!(12-k)!}$$

Comparison of SURE, ARIES, and CARE III for Example 31				
Parameters	Analytic solution	ARIES	CARE III	SURE bounds
$\lambda = 1 \times 10^{-4}$ $T = 8000$	5.28830×10^{-1}	5.28830×10^{-1}	5.28830×10^{-1}	$(5.28830 \times 10^{-1},\ 5.28830 \times 10^{-1})$

Example 32

Example 32 is a system of five components in which a majority of the components must be working properly in order for the system to operate. The system is not constrained to only critically coupled faults. In the model for this example, λ is the fault-arrival rate and δ is the fault-recovery rate.



Example 32: Five-plex system with permanent faults.

The unreliability of the system in example 32 for a given mission time T is

$$P_F(T) \approx \frac{3\lambda(1 - 5e^{-4\lambda T} + 4e^{-5\lambda T})}{\delta} + 5(1 - e^{-\lambda T})^4 e^{-\lambda T} + (1 - e^{-\lambda T})^5$$

Comparison of SURE, ARIES, and CARE III for Example 32				
Parameters	Analytic approximation	ARIES	CARE III	SURE bounds
$\lambda = 1 \times 10^{-4}$ $\delta = 3.6 \times 10^{3}$ $T = 10$	5.81686×10^{-12}	5.81907×10^{-12}	5.61405×10^{-10}	$(5.69942 \times 10^{-12},\ 5.83454 \times 10^{-12})$

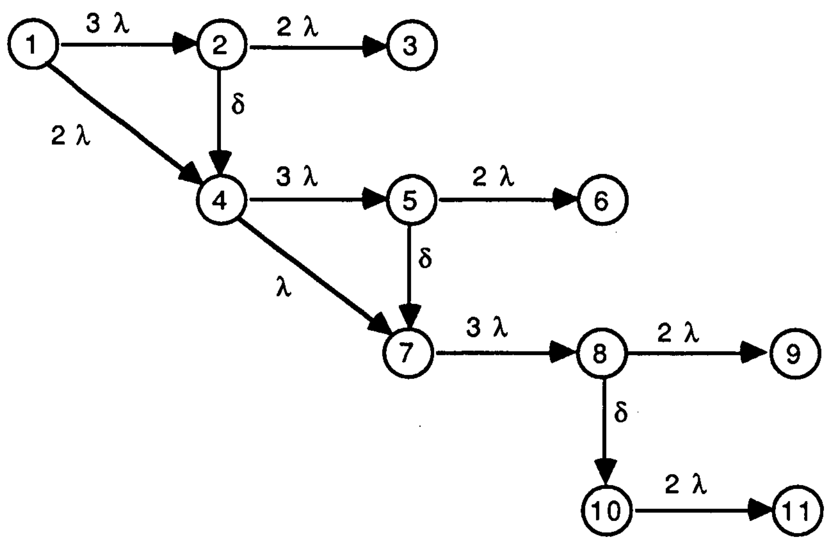
Note that CARE III's solution for example 32 is more than two orders of magnitude greater than the other solutions. The inaccuracy in this case is probably due to CARE III's inability to directly model critical-triple faults.

Example 33

Example 33 is a triad with two powered spares. In the model, λ is the fault-arrival rate and δ is the fault-recovery rate.

The following is an approximation for the unreliability of this system for a mission time T:

$$P_F(T) \approx \frac{6\lambda^2 T}{\delta} + 5(1 - e^{-\lambda T})^4 e^{-\lambda T} + (1 - e^{-\lambda T})^5$$



Example 33: Triad with two powered spares.

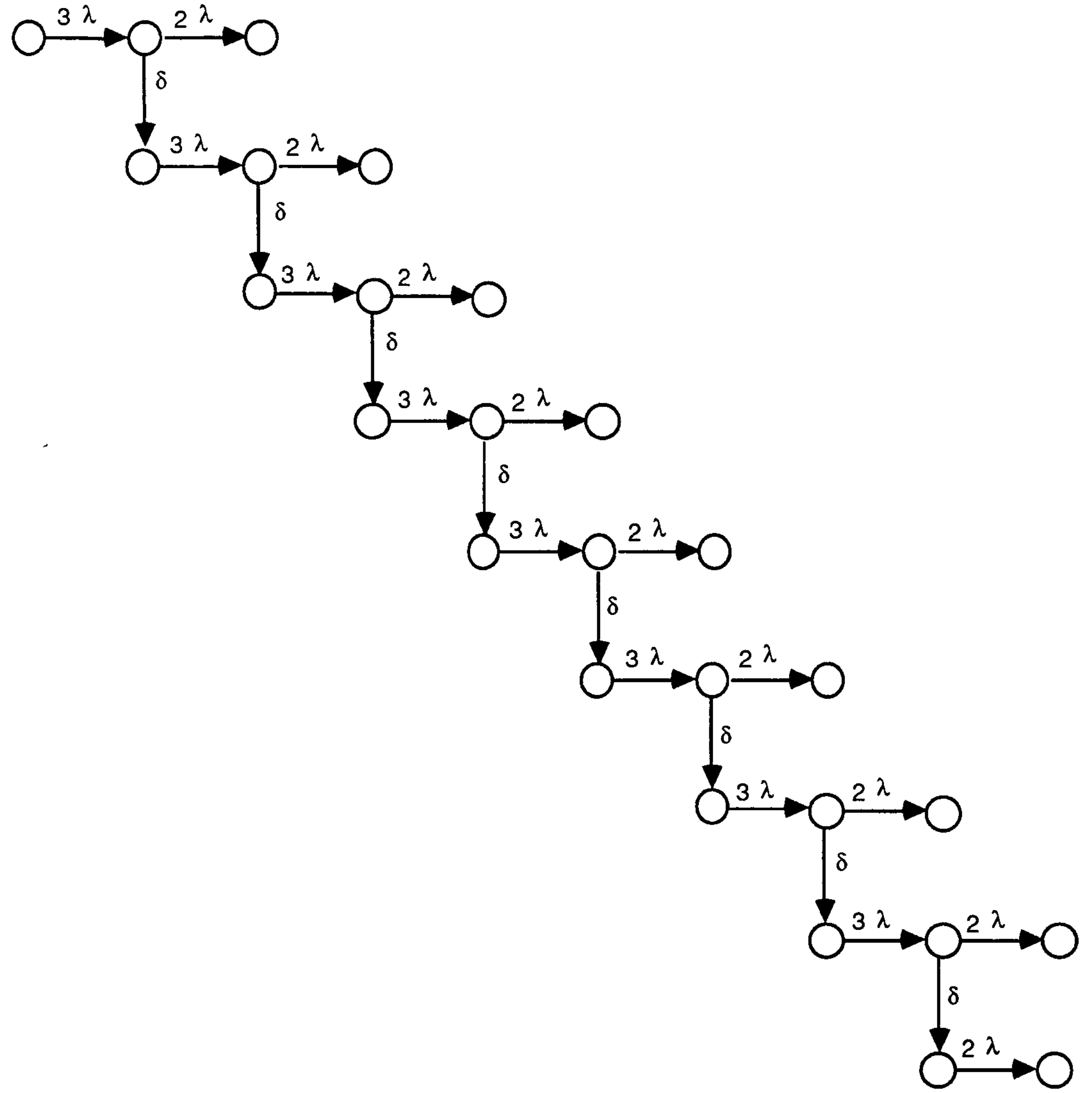
Comparison of SURE, ARIES, and CARE III for Example 33				
Parameters	Analytic approximation	ARIES	CARE III	SURE bounds
$\lambda = 1 \times 10^{-4}$ $\delta = 3.6 \times 10^{3}$ $T = 10$	1.71653×10^{-10}	1.71653×10^{-10}	1.70686×10^{-10}	$(1.69100 \times 10^{-10},\ 1.72084 \times 10^{-10})$

Example 34

Example 34 is a triad of components with seven unpowered spare components. The system must have two working components to keep the system operational. In the model, λ is the fault-arrival rate and δ is the fault-recovery rate.

This example involves unpowered spares; thus, CARE III cannot be used since it assumes that all spares are powered. The following is an approximation for the unreliability of the system:

$$P_F(T) \approx \frac{2(1 - e^{-3\lambda T})}{\delta} + 1 - 3^8 e^{-2\lambda T} + e^{-3\lambda T}\left[6560 + 2186(3\lambda T) + 728(3\lambda T)^2 + \frac{242(3\lambda T)^3}{6} + \frac{80(3\lambda T)^4}{24} + \frac{26(3\lambda T)^5}{120} + \frac{8(3\lambda T)^6}{720} + \frac{2(3\lambda T)^7}{5040}\right]$$



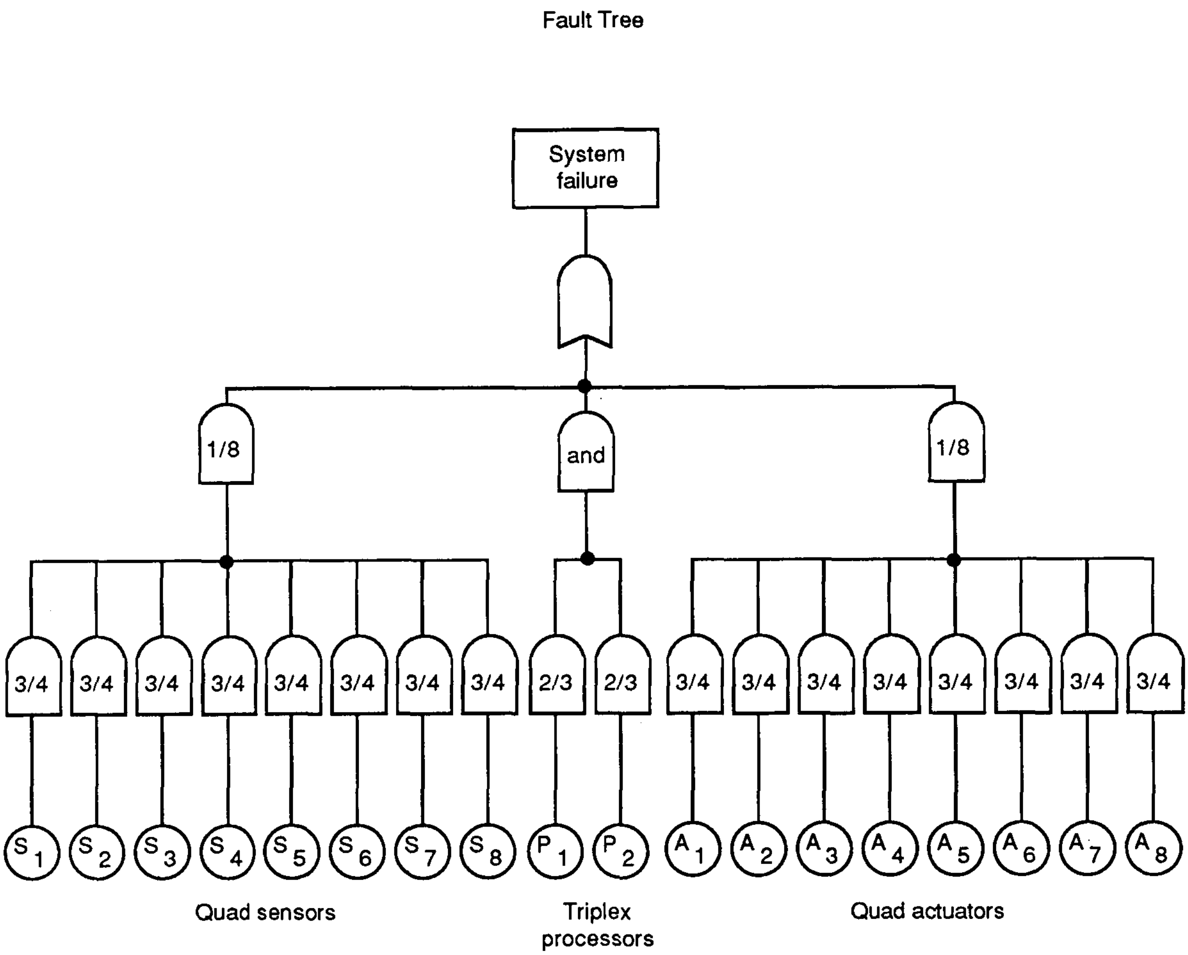
Example 34: Triad with seven unpowered spares.

Comparison of SURE and ARIES for Example 34			
Parameters	Analytic approximation	ARIES	SURE bounds
$\lambda = 1 \times 10^{-4}$ $\delta = 3.6 \times 10^{3}$ $T = 87600$	9.99839×10^{-1}	9.99×10^{-1}	$(9.99873 \times 10^{-1},\ 9.99887 \times 10^{-1})$

Example 35

The system for example 35 is a simple flight-control system composed of eight sets of quad sensors, eight sets of quad actuators, and two triplex processors. In this system only one triplex of the processors is performing critical functions, and this triplex is replaced by the second set once the first set experiences three faults. The system fails if a sensor set is lost, an actuator set is lost, or two of the processors in either triplex are lost. A sensor set is lost when three of the four sensors fail. Similarly, an actuator set is lost if three of the four actuators fail.

Since the Markov model for this system is extremely large, over 30 000 states, the fault-tree representation of this model is given in the figure. To generate the Markov model for this example, the ASSIST (Abstract Semi-Markov Specification Interface to the SURE Tool) program was used (ref. 15). To generate a model with ASSIST, the user inputs rules describing the behavior of the system being modeled using a predefined abstract language. From these rules, ASSIST automatically generates the semi-Markov model in the format needed as input to SURE.



Example 35: Fault tree for a flight-control system.

The following parameters were used for this model: λ_S is the failure rate of a sensor, λ_A is the failure rate of an actuator, and λ_P is the failure rate of a processor. The following is an approximation for the probability of this system's failure:

$$P_F(T) \approx 32\lambda_S^3 T^3 + 32\lambda_A^3 T^3 + 9\lambda_P^4 T^4$$

Comparison of SURE, ARIES, and CARE III for Example 35				
Parameters	Analytic approximation	ARIES	CARE III	SURE bounds
$\lambda_S = 1 \times 10^{-4}$ $\lambda_A = 1 \times 10^{-4}$ $\lambda_P = 1 \times 10^{-3}$ $T = 10$	1.54000×10^{-7}	1.50909×10^{-7}	1.50909×10^{-7}	$(1.50823 \times 10^{-7},\ 1.55344 \times 10^{-7})$

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15. Supplementary Notes

16. Abstract

This paper presents the results of an extensive effort to test the SURE (Semi-Markov Unreliability Range Evaluator) program. The SURE program is a reliability analysis tool developed at NASA Langley Research Center to compute upper and lower bounds on the death-state probabilities of semi-Markov models. These bounds are then used to assess the reliability of fault-tolerant computer systems modeled by semi-Markov processes. The SURE program was tested in three phases. In the first phase, several simple semi-Markov models were solved analytically for the exact death-state probabilities; then, these solutions were compared with the corresponding bounds given by SURE. During the second phase, much larger and more realistic models of fault-tolerant systems were examined. The SURE bounds for the unreliability of each of these models were compared with the unreliability estimates from other reliability analysis tools including CARE III (Computer-Aided Reliability Estimation, version 3), PAWS (Padé Approximation With Scaling), ARIES (Automated Reliability Interactive Estimation System), and STEM (Scaled Taylor Exponential Matrix). The final phase of testing consisted of an error analysis of the mathematical bounds implemented in the program.

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